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FINITE DIFFERENCE METHOD FOR STATIC AND DYNAMIC ANALYSIS: EULER-BERNOULLI BEAM ON WINKLER FOUNDATION AND VIBRATING MOTION OF SINGLE DEGREE OF FREEDOM SYSTEM

Gülçin TEKİN[1]*, Fethi KADIOĞLU[2]

[1*]Department of Civil Engineering, Yıldız Technical University, Davutpaşa Campus, Istanbul, Turkey
[2]Department of Civil Engineering, Istanbul Technical University, Ayazağa Campus, Istanbul, Turkey

E-mail: gulcint@yildiz.edu.tr [1], fkadioglu@itu.edu.tr [2]

Abstract: The Finite Difference Method (FDM) is one of the most powerful numerical solution techniques and has the ability to handle most types of analysis in structural mechanics. In this study, examples of physical modeling of vibrational motion of a single-degree-of-freedom-system (SDOF) under the effect of a harmonic external load are examined by considering the effects of different system parameters. In its simplest form, the problem is represented by a second order differential equation with constant coefficient. The relevant equation is solved analytically and at the same time, the compatibility of the results is tested using the finite-difference method. In addition, the analysis of the beam resting on the elastic foundation is considered. The analytical solution of the fourth order differential equation is obtained and the finite-difference method is used in order to obtain its numerical solution. Different numerical example problems are considered for the above mentioned two problems and the results are tested with the existing literature.

Keywords: *Finite-difference method, Beam-foundation interaction, Vibration, Numerical solution, Analytical Solution*

Introduction. In numerical analysis, the finite difference approximations of derivatives are simple, of rapid convergence and accurate to solve differential equations of which their analytical solutions are difficult or impossible to find. Finite difference methods (FDMs) transform partial differential equations (pdes) into a system of linear algebraic equations. In the FDMs, discrete approximations are used for the derivatives in the differential equation. These approximations are derived from the Taylor series expansions. Three of the approximations are Forward, Backward and Central differences. Commonly, the central difference formula is used due to fact that it yields better accuracy. The error for the central difference difference decreases quadratically as the step size decreases whereas for the forward/backward schemes the decrease is linear. The finite difference approximations relate the value of the dependent variable at a point in the solution region to its values at the neighboring points. The solution region is divided into n subintervals of length "h". In to get a good approximation, step size should be sufficiently small. Large step size increases simulation speed in practice, but create instabilities.

In engineering, beams are fundamental components and are widely used as an accurate and simple model for analysis of complex engineering structures. This study aims to investigate the analytic and the numerical solution of beam-type structures which are resting on an elastic foundation. For numerical solution, the FDM is used.

The first important studies on the behavior of elastic foundation presented by Winkler [1] in 1867 and as a result of these studies, Winkler hypothesis has been revealed. The Winkler model is most commonly used in practice, since the soil behavior is represented with a very simple approach. The general idea of the hypothesis is based on the fact that the foundation consists of infinitely close, elastic and linear springs. Under the effect of the uniformly distributed load, the ground reaction force is formulated briefly as follows:

$$q(z) = -kv(z)$$

where the spring coefficient “k” is known as the subgrade reaction coefficient. According to this hypothesis, the foundation reaction at any point of the elastic and prismatic beam under the influence of various loads is proportional to the deflection value at the same point of the beam under consideration. Here, q(z) is the reaction of the elastic foundation and v(z) is the displacement in the vertical direction. Assumption of the Winkler hypothesis is that a force acting on the foundation causes deformation only at the point where it acts. In other words, Winkler considered the elastic foundation as a system of vertical springs that are not affected by each other, are infinitely close to each other and can move freely by compression. In Winkler's model, the only parameter that shows the character of the foundation is the parameter “k”. For this reason, the Winkler model is also referred to as a single parameter foundation model.

Many beam theories have been developed based on various assumptions. The simplest and the most commonly used by researchers is named as Euler-Bernoulli beam theory with the following kinematic assumptions: the cross-section is infinitely rigid in its own plane, the cross-section of a beam remains plane and normal to the deformed axis of the beam after deformation.

A large number of studies can be found in the literature on the analysis of beams with various theories and geometries. However, during the past two decades, the researcher’s attention has been drawn increasingly to the beam-foundation interaction problems. Develi [2] has investigated the vibration problem for a finite length of Timoshenko beam on Vlasov foundation and Winkler foundation. In this study, the elastic curve function is obtained from the differential equations of the beam. Comparison is made for Timoshenko beam and Euler-Bernoulli beam and it is observed that the displacement, shear force and bending moment values are close to each other. Eisenberger and Bielak [3] have considered externally loaded free-end beams on a two-parameter elastic foundation. It has been observed that the interaction with foundation depends on the beam length, the bending stiffness of the beam and the foundation stiffness parameters. Ike [4] considered Euler-Bernoulli beams on Winkler foundation by the point collocation method. The beams subjected to uniformly distributed loads are considered. It is observed that the values of deflection and bending moment at the mid-point of the beam decreased with the increase of the subgrade reaction modulus. Doğan [5] has examined the homogeneous and non-homogeneous conditions of foundation for weightless beams. In the numerical solutions, different loading types and foundation coefficients are considered. Sign changes for vertical displacement are observed in the samples examined for tension and compression conditions. Karamahmutoğlu [6] focused on the analysis of sheet- pile walls and beams on elastic foundation by using the finite-difference method. The Winkler foundation model is used as the foundation model. The results are compared with the available examples in the literature. Heteyni [7] worked on the Winkler foundation model. In this study, deflection and bending moment values are obtained for different points of finite and infinite beams under different loading conditions.

A further analysis is considered to discuss the dynamic response of a linear single-degree-of-freedom (SDOF) oscillator. Structures respond to earthquake excitation as either simple or complex oscillators. SDOF systems are used to represent the simple oscillators whereas multi-degree-of-freedom (MDOF) systems are used to represent complex oscillators.The motion of the linear SDOF system under harmonic load is solved analytically and also the FDM is used to determine the displacement response time histories of linear SDOF systems. Various studies in the earthquake engineering literature devoted to the dynamic response analysis of systems under different types of dynamic loads [8-12].

Methodology.

Central Finite-Difference Method

The Taylor Series of a real or complex f(z) function with any order derivative, in the range ($a-r$, $a+r$), where ‘a’ is a real or complex number, is defined as:

$$f(a) + f'(a)(z-a) + \frac{f^{(2)}(z)}{2!}(z-a)^2 + \ldots + \frac{f^{(n)}(z)}{n!}(z-a)^n \tag{1}$$

If f(z) function is expanded using the Taylor Series at the points $z_{i-1} = z_i - \Delta z$ and $z_{i+1} = z_i + \Delta z$, the following equations are obtained:

$$f(z_{i-1}) = f(z_i) - f'(z_i)\Delta z + \frac{f^{(2)}(z_i)}{2!}(\Delta z)^2 - \frac{f^{(3)}(z_i)}{3!}(\Delta z)^3 + \frac{f^{(4)}(z_i)}{4!}(\Delta z)^4 + \ldots \tag{2}$$

$$f(z_{i+1}) = f(z_i) + f'(z_i)\Delta z + \frac{f^{(2)}(z_i)}{2!}(\Delta z)^2 + \frac{f^{(3)}(z_i)}{3!}(\Delta z)^3 + \frac{f^{(4)}(z_i)}{4!}(\Delta z)^4 + \ldots \tag{3}$$

These expressions can be arranged as follows:

$$f_{i-1} = f_i - f_i'\Delta z + \frac{f_i^{(2)}}{2!}(\Delta z)^2 - \frac{f_i^{(3)}}{3!}(\Delta z)^3 + \frac{f_i^{(4)}}{4!}(\Delta z)^4 + \ldots \tag{4}$$

$$f_{i+1} = f_i + f_i'\Delta z + \frac{f_i^{(2)}}{2!}(\Delta z)^2 + \frac{f_i^{(3)}}{3!}(\Delta z)^3 + \frac{f_i^{(4)}}{4!}(\Delta z)^4 + \ldots \tag{5}$$

Subtracting (4) from (5) and considering the first three terms on the right, the first derivative expression for the z_i point is obtained as follows:

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2\,\Delta z} \tag{6}$$

Again, if the expressions (4) and (5) are added side by side and the first three terms are considered, the second derivative expression for the z_i point is obtained as follows:

$$f_i^{(2)} = \frac{f_{i-1} - 2f_i + f_{i+1}}{(\Delta z)^2} \tag{7}$$

Let's expand the funciton f(z) into the series at the points $z_{i-2} = z_i - 2\Delta z$ and $z_{i+2} = z_i + 2\Delta z$:

$$f(z_{i-2}) = f(z_i) - f'(z_i)\,2\Delta z + \frac{f^{(2)}(z_i)}{2!}(2\Delta z)^2 - \frac{f^{(3)}(z_i)}{3!}(2\Delta z)^3 + \frac{f^{(4)}(z_i)}{4!}(2\Delta z)^4 + \ldots \tag{8}$$

$$f(z_{i+2}) = f(z_i) + f'(z_i)\,2\Delta z + \frac{f^{(2)}(z_i)}{2!}(2\Delta z)^2 + \frac{f^{(3)}(z_i)}{3!}(2\Delta z)^3 + \frac{f^{(4)}(z_i)}{4!}(2\Delta z)^4 + \ldots \tag{9}$$

The arranged version of these expressions is:

$$f_{i-2} = f_i - f_i'\,2\Delta z + \frac{f_i^{(2)}}{2!}(2\Delta z)^2 - \frac{f_i^{(3)}}{3!}(2\,\Delta z)^3 + \frac{f_i^{(4)}}{4!}(2\Delta z)^4 + \ldots \tag{10}$$

$$f_{i+2} = f_i + f_i'\,2\Delta z + \frac{f_i^{(2)}}{2!}(2\Delta z)^2 + \frac{f_i^{(3)}}{3!}(2\Delta z)^3 + \frac{f_i^{(4)}}{4!}(2\Delta z)^4 + \ldots \tag{11}$$

The expression (4) is multiplied by (-2), and the expression (5) is multiplied by (+2) and added side by side and considering the first five terms:

$$-2f_{i-1} + 2f_{i+1} = 4f_i' + 4\frac{f_i^{(3)}}{3!}(\Delta z)^3 \tag{12}$$

The expression (10) is multiplied by (-1), and the expression (11) by (+1) and summed side by side and considering the first five terms:

$$-f_{i-2}+f_{i+2}=2f_i'(2\Delta z)+2\frac{f_i^{(3)}}{3!}(2\Delta z)^3 \tag{13}$$

If (12) is subtracted from the expression (2.13), the third derivative expression for the z_i point is obtained as follows:

$$f_i^{(3)}=\frac{-f_{i-2}+2f_{i-1}-2f_{i+1}+f_{i+2}}{2(\Delta z)^3} \tag{14}$$

If (4) and (5) expressions are multiplied by (+4) and summed side by side and the first five terms are considered:

$$4f_{i-1}+4f_{i+1}=8f_i+8\frac{f_i^{(2)}}{2!}(\Delta z)^2+8\frac{f_i^{(4)}}{4!}(\Delta z)^4 \tag{15}$$

The expressions (10) and (11) are summed side by side and the first five terms are considered:

$$f_{i-2}+f_{i+2}=2f_i+2\frac{f_i^{(2)}}{2!}(2\Delta z)^2+8\frac{f_i^{(4)}}{4!}(2\Delta z)^4 \tag{16}$$

If (16) is subtracted from (15), the fourth derivative expression for the z_i point is obtained as follows:

$$f_i^{(4)}=\frac{f_{i-2}-4f_{i-1}+6f_i-4f_{i+1}+f_{i+2}}{(\Delta z)^4} \tag{17}$$

Analytical Solution of Euler-Bernoulli Beam on Elastic Foundation

The fourth-order differential equation for the Euler-Bernoulli beam on the Winkler foundation (see Figure 1) can be expressed as:

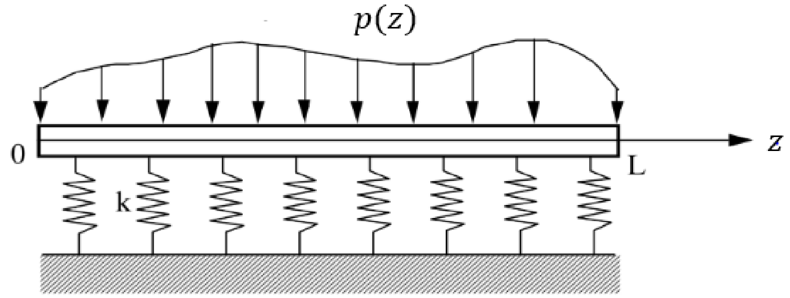

Fig. 1. Beam–elastic foundation interaction

$$EI_x\frac{d^4v}{dz^4}+kv(z)=p(z) \tag{18}$$

where *E* is the modulus of elasticity (Young's modulus) of the beam material, *Ix* is the moment of inertia of the section about the x-axis, *k* is the subgrade reaction coefficient (Winkler's modulus), *p*(z) is the distributed load applied to the beam.

Firstly, the homogeneous solution of the differential equation is obtained as follows:

$$v_h(z)=e^{\beta z}\left[c_1\cos(\beta z)+c_2\cos(\beta z)\right]+e^{-\beta z}\left[c_3\cos(\beta z)+c_4\sin(\beta z)\right] \tag{19}$$

where c_1, c_2, c_3 and c_4 are integration constants and β is a problem constant in units of (1/m) calculated from the following relation:

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} \tag{20}$$

The particular solution depends on the load and it is given as follows for the uniformly distributed load:

$$v_p \;\; z \;\; = \frac{p}{k} \tag{21}$$

Now the general solution is the sum of the homogeneous and particular solutions as follows:

$$v_g \;\; z \;\; = \frac{p}{k} + e^{\beta z}\left[c_1 \cos \;\; \beta z \;\; + c_2 \cos \;\; \beta z \;\; \right] + e^{-\beta z}\left[c_3 \cos \;\; \beta z \;\; + c_4 \sin \;\; \beta z \;\; \right] \tag{23}$$

In order to determine the constants c_1, c_2, c_3 and c_4, it is necessary to use to the boundary conditions of the beam. In a simply supported finite beam with a length of L, the boundary conditions can be written as: v(z=0) = 0, v(z=L) = 0 and $v^{(2)}$(z=0) = 0, $v^{(2)}$(z=L) = 0. If the boundary conditions are adapted to Equation (2.23), the integration constants c_1, c_2, c_3 and c_4 can be obtained as follows:

$$c_1 = -\frac{p\left[\sin \;\; \beta L \;\; ^2 - \cos \;\; \beta L \;\; + \cos \;\; \beta L \;\; e^{\beta L} - \cos \;\; \beta L \;\; e^{-\beta L} + \cos \;\; \beta L \;\; ^2 e^{-2\beta L} + \sin \;\; \beta L \;\; ^2 e^{-2\beta L}\right]}{k\left[\cos \;\; \beta L \;\; ^2 e^{-2\beta L} + \cos \;\; \beta L \;\; ^2 e^{2\beta L} + \sin \;\; \beta L \;\; ^2 e^{-2\beta L} + \sin \;\; \beta L \;\; ^2 e^{2\beta L} - 2\cos \;\; \beta L \;\; ^2 + 2\sin \;\; \beta L \;\; ^2\right]} \tag{24}$$

$$c_2 = -\frac{p\left[\sin \;\; \beta L \;\; e^{\beta L} - 2\cos \;\; \beta L \;\; \sin \;\; \beta L \;\; + \sin \;\; \beta L \;\; e^{-\beta L}\right]}{k\left[\cos \;\; \beta L \;\; ^2 e^{-2\beta L} + \cos \;\; \beta L \;\; ^2 e^{2\beta L} + \sin \;\; \beta L \;\; ^2 e^{-2\beta L} + \sin \;\; \beta L \;\; ^2 e^{2\beta L} - 2\cos \;\; \beta L \;\; ^2 + 2\sin \;\; \beta L \;\; ^2\right]} \tag{25}$$

$$c_3 = -\frac{p\left[\sin \;\; \beta L \;\; ^2 - \cos \;\; \beta L \;\; ^2 - \cos \;\; \beta L \;\; e^{\beta L} + \cos \;\; \beta L \;\; e^{-\beta L} + \cos \;\; \beta L \;\; ^2 e^{2\beta L} + \sin \;\; \beta L \;\; ^2 e^{2\beta L}\right]}{k\left[\cos \;\; \beta L \;\; ^2 e^{-2\beta L} + \cos \;\; \beta L \;\; ^2 e^{2\beta L} + \sin \;\; \beta L \;\; ^2 e^{-2\beta L} + \sin \;\; \beta L \;\; ^2 e^{2\beta L} - 2\cos \;\; \beta L \;\; ^2 + 2\sin \;\; \beta L \;\; ^2\right]} \tag{26}$$

$$c_4 = -\frac{p\left[\sin \;\; \beta L \;\; e^{\beta L} - 2\cos \;\; \beta L \;\; \sin \;\; \beta L \;\; + \sin \;\; \beta L \;\; e^{-\beta L}\right]}{k\left[\cos \;\; \beta L \;\; ^2 e^{-2\beta L} + \cos \;\; \beta L \;\; ^2 e^{2\beta L} + \sin \;\; \beta L \;\; ^2 e^{-2\beta L} + \sin \;\; \beta L \;\; ^2 e^{2\beta L} - 2\cos \;\; \beta L \;\; ^2 + 2\sin \;\; \beta L \;\; ^2\right]} \tag{27}$$

When these constants are substituted into Equation (2.23) and simplified, the deflection at the mid-point of the beam obtained as follows:

$$v_m = \frac{p\left[1 - \dfrac{2\cos\left(\dfrac{\beta L}{2}\right)\cosh\left(\dfrac{\beta L}{2}\right)}{\cos(\beta L) + \cos(\beta L)}\right]}{k} \tag{28}$$

The bending moment at the mid-point of the beam is obtained as follows:

$$M_m = \frac{p}{2\beta^2}\frac{\sinh\left(\dfrac{\beta L}{2}\right)\sin\left(\dfrac{\beta L}{2}\right)}{\cos(\beta L) + \cos(\beta L)} \tag{29}$$

Analytical Solution of Forced Vibration of a SDOF

The mass "m (in kg)" shown in Figure 2, is connected to both spring and the dashpot, the reaction against the external force on the spring increases proportional to the displacement of the body from the equilibrium position, meanwhile the reaction at the dashpot increases with the velocity of the mass.

The second-order differential equation for the equation of motion can be expressed as:

$$m\ddot{x}+c\dot{x}+kx=F(t) \tag{30}$$

In order to obtain the general solution of the differential relation, firstly the homogenous solution must be obtained. Let's rewrite equation (30) by making right hand side equal to zero. For the homogenous solution, the below exponential function is proposed:

$$x=e^{\lambda t} \tag{31}$$

Substituting this expression and its derivatives yields:

$$e^{\lambda t}\left(m\lambda^2+c\lambda+k\right)=0 \tag{32}$$

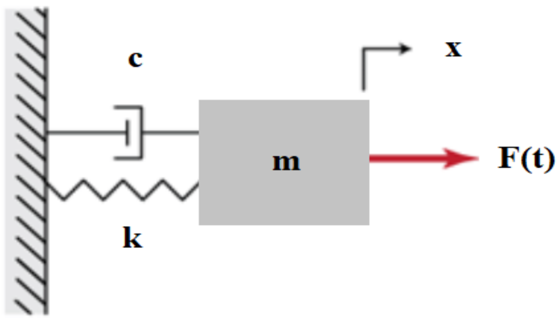

Fig. 2. Mass-spring-damper system

where F(t) is a time-dependent force in (N), k is the spring constant in (N/m), and c is the coefficient of the dashpot in (Ns/m).

Since $e^{\lambda t}$ can never be zero, a solution is possible provided $m\lambda^2+c\lambda+k=0$. Hence, two values of λ are:

$$\lambda_{1,2}=\frac{-c\pm\sqrt{c^2-4mk}}{2m}=-\frac{c}{2m}\pm\sqrt{\frac{c^2}{4m^2}-\frac{k}{m}} \tag{33}$$

Here the value which makes radical in (2.33) equal to zero is called critical damping coefficient, c_c.

$$c_c=\sqrt{4mk}=2\sqrt{mk}=2m\omega_n \tag{34}$$

where ω_n is the natural circular frequency of vibration; its units are [rad/sec]. There are 3 possible combinations of λ_1 and λ_2 which must be considered. Here, ξ is defined as a damping ratio that is given as:

$$\xi=\frac{c}{c_c} \tag{35}$$

The cases of damping are categorized via ξ ;

- $\xi>1$ or $c>c_c$: Overdamped motion. The system turns back to its original position without oscillating.

- ξ=1 or c=cc : Critically damped motion. The system shows tendencies to come to equilibrium as quick as possible without oscillating.
- ξ<1 or c<cc: Underdamped motion. The system oscillates with a gradual decrements to zero.

Most engineering structures fall into this category c<cc. In this study, the system is considered as "underdamped". By considering a relationship between the mass, stiffness and damping ratio, revised versions of the equations in terms of the natural circular frequency and damping ratio can be formed as follows:

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \tag{36}$$

and

$$\lambda_{1,2} = -\xi\omega_n \pm \sqrt{(\xi\omega_n)^2 - (\omega_n)^2} \tag{37}$$

or

$$\lambda_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \tag{38}$$

Underdamped condition corresponds to the negative value of the quantity inside the square root which means that:

$$\lambda_{1,2} = -\xi\omega_n \pm i\omega_n\sqrt{1-\xi^2} \tag{39}$$

where, i is the complex number. And, $\omega_n\sqrt{1-\xi^2} = \omega_d$.

Substituting $\lambda_{1,2}$ values into $x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ where C_1 and $C2$ are arbitrary constants, the solution is:

$$x = e^{-\xi\omega_n t}\left(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right) \tag{40}$$

Equation (2.40) can be written as:

$$x = e^{-\xi\omega_n t}\left[\cos(\omega_d t)(C_1 + C_2) + \sin(\omega_d t)(iC_1 - iC_2)\right] \tag{41}$$

by using $e^{i\theta} = \cos\theta + i\sin\theta$. If arbitrary constants are renamed as $B_1 = C_1 + C_2$ and $B_2 = iC_1 - iC_2$, homogenous part of the solution has become:

$$x_h = B_1 e^{-\xi\omega_n t}\cos(\omega_d t) + B_2 e^{-\xi\omega_n t}\sin(\omega_d t) \tag{42}$$

Homogenous solution will die out after a period of time due to friction. Let's suggest an external harmonic force has a form of $F(t) = F_0 \sin(\Omega t)$. Here Ω is the angular velocity of mentioned force. The form of particular solution is,

$$x_p = A\sin(\Omega t) + B\sin(\Omega t) \tag{43}$$

By considering time derivatives of (2.43) and substituting $x_p, \dot{x}_p, \ddot{x}_p$ into the Equation (30) results in:

$$(-Am\Omega^2 - cB\Omega + kA)\sin(\Omega t) + (-Bm\Omega^2 - cA\Omega + kB)\cos(\Omega t) = F_0 \sin(\Omega t) \tag{44}$$

The coefficients of $\sin(\Omega t)$ and $\cos(\Omega t)$ on each side of the equation must be equal to each other as follows:

$$-Am\Omega - cB\Omega + kA = F_0 \tag{45}$$

$$-Bm\Omega^2 + cA\Omega + kB = 0 \tag{46}$$

As a result:

$$A = \frac{F_0\left(k - m\Omega^2\right)}{\left(k - m\Omega^2\right)^2 + \left(\Omega c\right)^2} \tag{47}$$

$$B = \frac{-F_0\Omega c}{\left(k - m\Omega^2\right)^2 + \left(\Omega c\right)^2} \tag{48}$$

Recall, $x_g = x_h + x_p$

$$x(t) = B_1 e^{-\xi\omega_n t}\cos(\omega_d t) + B_2 e^{-\xi\omega_n t}\sin(\omega_d t) + A\sin(\Omega t) + B\cos(\Omega t) \tag{49}$$

As A and B parameters can be determined from known values, only unknowns are B_1 and B_2. Let initial conditions of the differential equation are $x(0) = x(0)$, and $\dot{x}(0) = \dot{x}(0)$

$$B_1 = x(0) - B \tag{50}$$

$$B_2 = \frac{\dot{x}(0) + \left[x(0) - B\right]\xi\omega_n - A\Omega}{\omega_d} \tag{51}$$

This is the general solution of the forced vibration of SDOF systems.

Numerical Examples and Discussion.

In this section, the first two examples are considered for the Euler-Bernoulli beam-elastic foundation interaction problems. And the last two examples are considered in order to discuss the effects of different parameters of the system on the vibration behavior. In order to show the efficiency of the FDM numerical solution technique, the analytical results are compared with the results of FDM.

Example 1

Vertical displacement and bending moment values at the mid-point of a simply supported beam under uniformly distributed load and resting on an elastic foundation are calculated analytically and the results are compared with the reference study in the literature and the results of FDM.

Properties of the cross-section of the beam;

$EI_x = 1{,}89 \times 10^6$ kNm^2

Foundation parameter;

$k = 6{,}92 \times 10^3$ t/m^2

The comparison results of the deflection and bending moment with the reference study [4] and the FDM are presented in Table 1. In order to use the central finite difference method for the solution, the beam is divided into four equal sub-intervals. By considering the boundary conditions of the simply supported beam, the sets of algebraic equations are constructed and unknowns are calculated. The maximum deflection value is also calculated by using the FDM. As seen from the following table, the results are close enough. The FDM gives satisfactory results with using large step size and saving time.

Table 1. The values of deflection and bending moment for the mid-point of the beam resting on elastic foundation

	v_{max} [m]	M_{max} [kNm]
Reference Study [4]	$1{,}36 \times 10^{-3}$	23,9
Analytical Result	$1{,}42 \times 10^{-3}$	23,7
FDM	$1{,}20 \times 10^{-3}$	

Example 2

The variation of the deflection values at the midpoint for different subgrade reaction coefficients is discussed. The simply supported beam under the effect of uniformly distributed load resting on the elastic foundation is considered.

Properties of the cross-section of the beam;

$EI_x = 1{,}89 \times 10^6$ kNm^2

Foundation parameters;

$k_1 = 6 \times 10^3$ t/m^2 $k_2 = 12 \times 10^3$ t/m^2 $k_3 = 25 \times 10^3$ t/m^2

Midpoint deflection values are calculated for three different foundation parameters and the results are shown in Table 2. As expected, the deflection values decreased as the foundation stiffness increased.

Table 2. Deflection values for different coefficient of subgrade reaction

k (t/m²)	6000	12000	25000
$v_{max}(m)$	$1{,}59 \times 10^{-3}$	$0{,}89 \times 10^{-3}$	$0{,}45 \times 10^{-3}$

Example 3

The external harmonic force $F(t) = 100\cos(11t)$ N is applied to a stationary $150\ kg$ block is attached to the wall with the spring which has a $15\ kN/m$ spring constant and also connected to a dashpot which has $150\ Ns/m$ damping coefficient. Show graphically the convergence of FDM at the first 10 seconds for different time intervals as: $\Delta t = 0.02$, $0.005\ s$ and $\Delta t = (0.1\ T) = 0.06283\ s$.

Initial conditions are, $x(0) = \dot{x}(0) = 0$

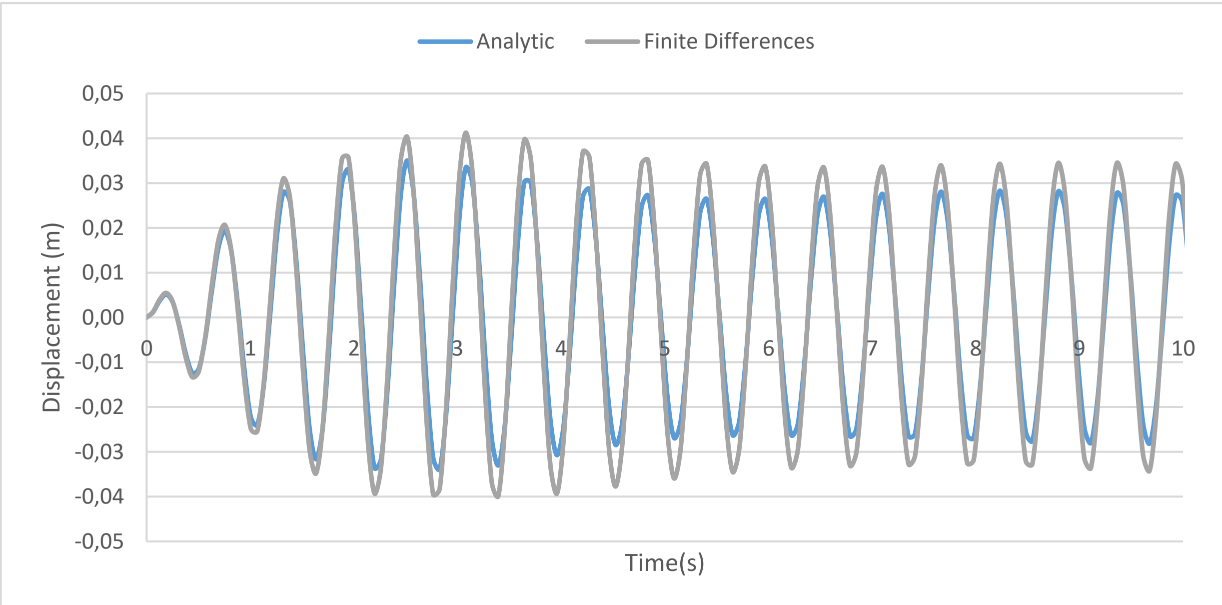

Fig. 3. Outputs of numerical solution versus analytic solution for $\Delta t = 0.1\ T$

It is obvious in Figure 3 that $\Delta t = 0.1\ T$ is not enough and time interval must be decreased to have better results. Now the analysis is performed for time interval $\Delta t = 0.02\ s$ and it is expected that the numerical analysis results to be more precise.

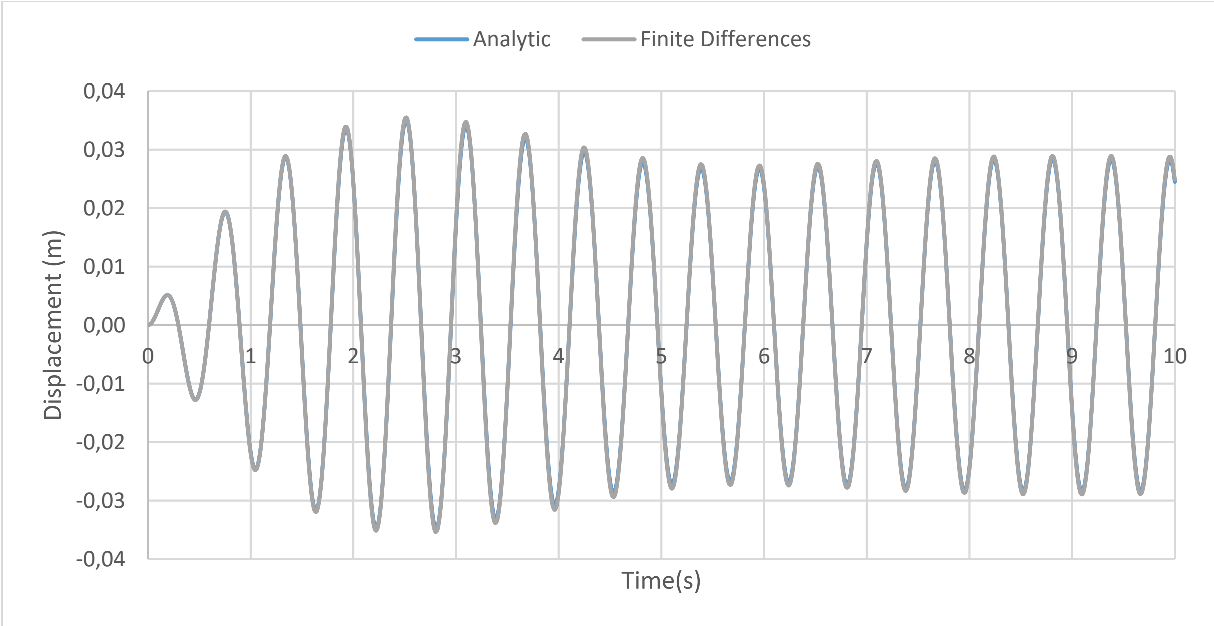

Fig. 4. Outputs of numerical solutions versus analytic solution for $\Delta t = 0.02\ s$

When compared to $\Delta t = 0.1\ T$, we can say that the obtained displacements are quite precise as expected.

Although the analytical method gives the result exactly, it is understood that a numerical analysis can lead us to almost the same result performing simple analysis.

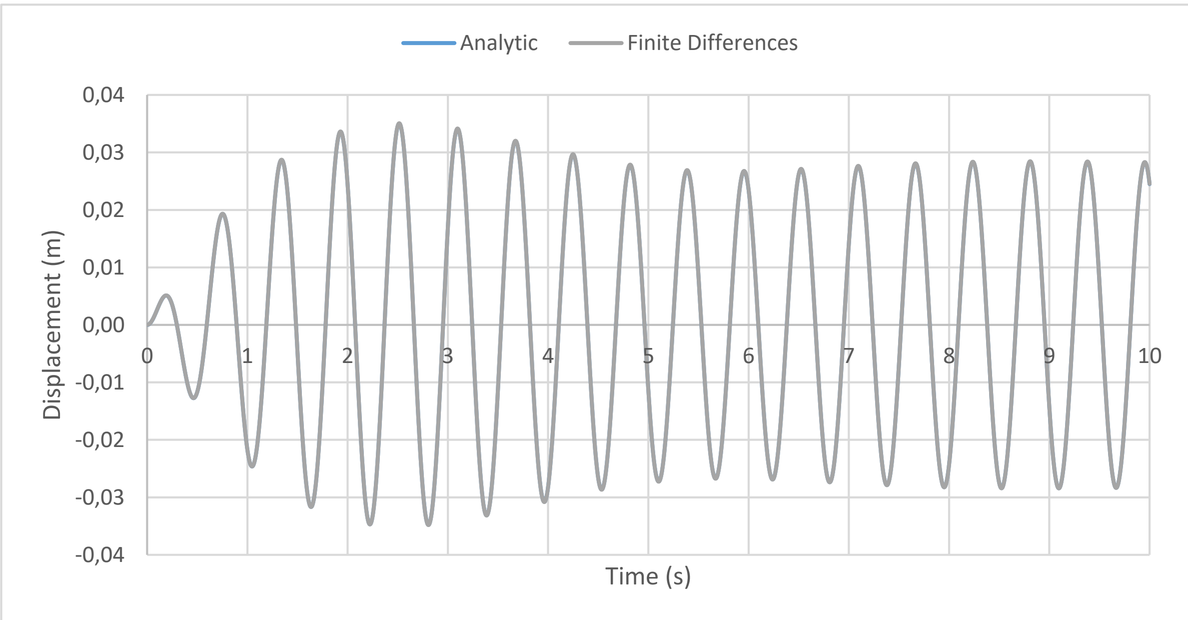

Fig. 5. Outputs of numerical solutions versus analytic solution for $\Delta t = 0.005\ s$

Example 4

This example is considered in order to examine the effect of change in system period (*T*=1,1.5 and 2 s.) on the behavior of the vibrating system. In order to change the system's period, mass is going to be constant, but spring constant will change.

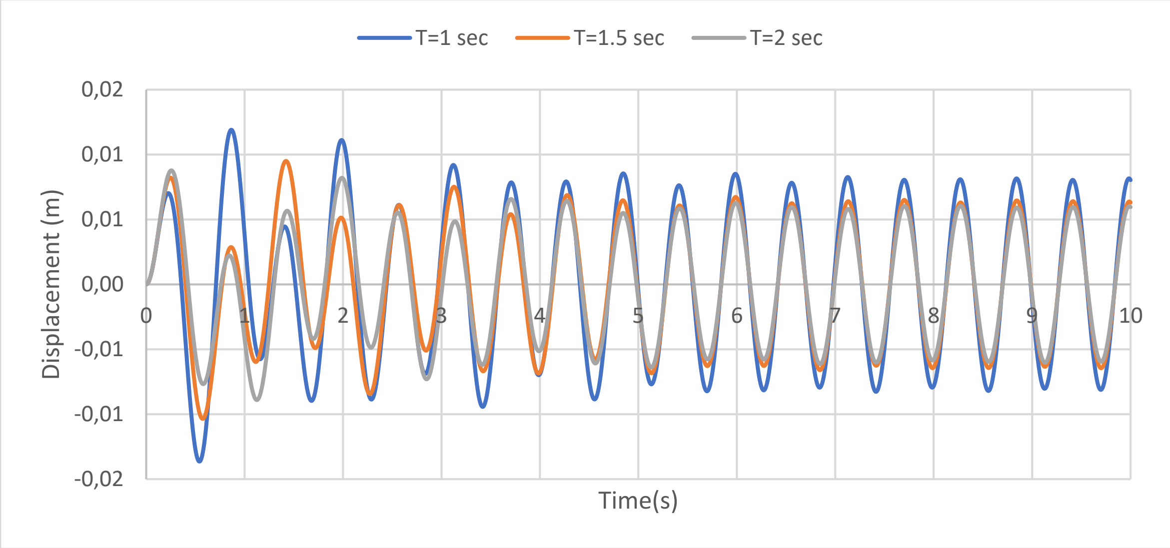

Fig. 6. Change of displacement for T=1, T=1.5 and T=2 seconds

As we expected, when the period increases, displacements are decreased.

Example 5

In this example, the behaviour of vibration of the system is analysed for different damping ratios as $\xi = 0.03$, $\xi = 0.5$ and $\xi = 0.9$. Let $\xi < 1$, as we investigated for underdamped system. Here, only the parameter c will be changed.

As the damping ratio increases, the amplitude of the vibration motions is decreased. In accordance with the definition of damping, increasing the damping coefficient leads to faster damping of the system and decreased displacement values.

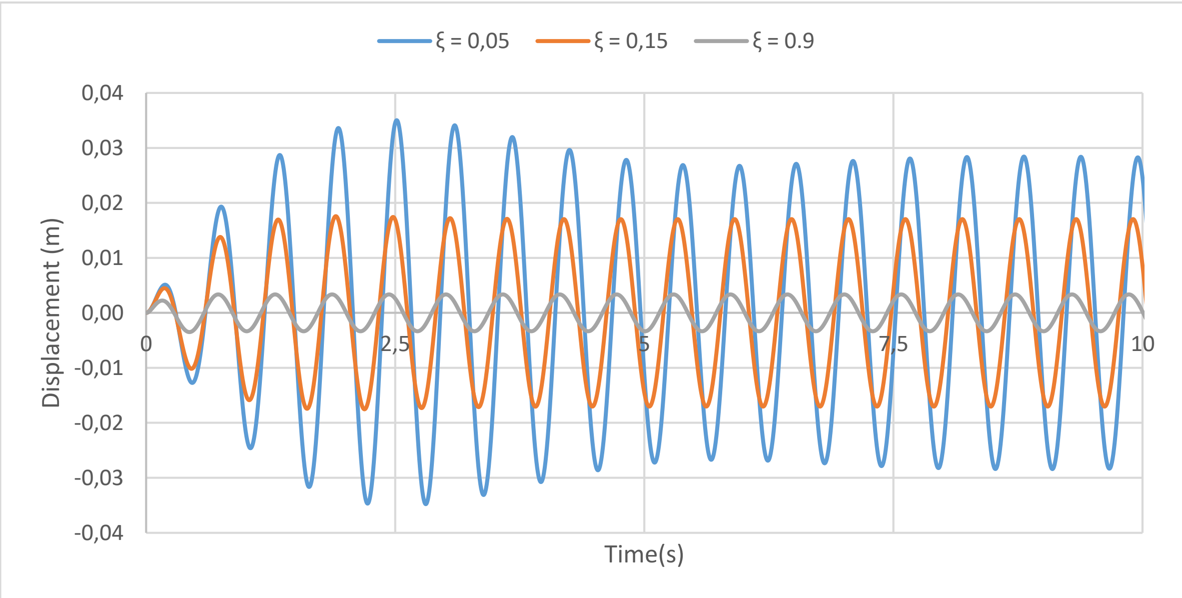

Fig. 7. Effect of different damping ratios as ξ = 0.05, ξ = 0.15 and ξ = 0.9

Conclusion. In this study, Euler-Bernoulli beam resting on elastic foundation is analyzed analytically and numerically by using the FDM. The maximum deflection and bending moment values are calculated for uniformly loaded simply supported beam and the results are compared with the literature. Sufficiently convergent results are obtained. In addition, the deflection values at the mid-point of the beam are found for different subgrade modulus. As expected, when the subgrade reaction coefficient increased, the vertical displacement value at the midpoint of the beam decreased. In addition, vibrational motion analysis of a SDOF system under the effect of a harmonic external load is examined by considering the effects of different system parameters. An analytical solution of the problem is obtained in addition to the its numerical solution. In order to get a good approximation in FDM, step size should be sufficiently small. When the period of the system increases due to the change of the spring constant, the displacements are decreased as expected. In addition, it is observed that the displacements decrease in case the damping ratio increases.

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