MACHINE SCIENCE

ISSN: 2227-6912
E-ISSN: 2790-0479

2023, № 1 Pages 68 – 75

# INVESTIGATION OF OPTIMAL CONTROL OF VARIABLE SYSTEMS IN THE DYNAMIC SPECTRUM

**Victor ARTEMYEV[1*], Artem MEDVEDEV[2], Victor YAROSHEVICH[3]**

[1*,2]Federal State Budgetary Educational Institution of Higher Education "Russian Biotechnological University (ROSBIOTECH)", Moscow, Russian Federation
[3]Belarusian State University of Economics, Minsk, Republic of Belarus

*E-mail:* artemyevvs@mgupp.ru[1*], am@helpexcel.pro[2], jaroshevich1970@gmail.com[3]

**Abstract:** The article presents a study of systems of difference equations, their role and application in modeling and analysis of dynamical systems in various fields. The purpose of the work is to analyze and develop methods for solving difference equations with a special emphasis on their application in automation and mechanics.

Difference equations are mathematical models describing the evolution of variables in a discrete time space. Their study allows for a deeper understanding of complex dynamic processes and effective modeling of systems of diverse nature.

The paper presents an overview of various methods of analysis and solution of difference equations. Numerical integration methods, approximation methods and stabilization methods of systems of difference equations are considered. Special attention is paid to the properties of stability, convergence and controllability of such systems.

The results of the study contribute to a deep understanding of the dynamics of systems of difference equations and the development of effective numerical methods for their solution. The solution of difference equations is widely used in automatic control, robotics, aviation, mechatronics and other industries. It plays an important role in modeling and controlling various systems, including electromechanical systems, robots, autopilots, automation processes, and others.

***Keywords:*** *numerical integration methods, approximation methods, stabilization methods for systems of difference equations.*

**Introduction.** Numerical integration methods play an important role in solving differential and difference equations, which are widely used in various fields of science and engineering. One of the most common methods is the Euler method, based on the approximation of the derivative by a difference ratio. This method allows you to approximate the solution of the equation by breaking it into small steps and sequentially updating the values of variables. Thus, the Euler method provides a fast and simple numerical solution of differential and difference equations. It allows you to approximate the value of the function at the next point based on the known value at the previous point. The Euler method is simple to implement, but may not be accurate enough for some tasks [1].

More precise numerical integration methods include Runge-Kutta methods, which use several intermediate steps to refine the approximate solution. Runge-Kutta methods of various orders of accuracy allow to achieve high accuracy of calculations with a sufficiently small integration step [2].

Another common method of numerical integration is the trapezoid method, which is based on the approximation of the area under the graph of the function. It represents the arithmetic mean of the function values at two adjacent points. The trapezoid method has high accuracy and is widely used in numerical calculations.

There are also numerical integration methods based on interpolation, splines and other mathematical approximations. These methods allow you to more flexibly and accurately approximate functions, taking into account their features and the requirements of a specific task.

Numerical integration methods play an important role in solving various problems, such as modeling physical processes, numerical solution of differential equations, calculation of integrals and other mathematical operations.

Approximation methods provide tools for the approximate representation of complex functions that cannot always be accurately described analytically. These methods include various approaches such as polynomial approximation, interpolation, spline approximation, least squares methods and others.

Polynomial approximation is one of the simplest and most widely used methods [3]. It is based on the representation of a function in the form of a polynomial that best approximates its values at a given interval. Depending on the degree of the polynomial, different degrees of approximation accuracy can be achieved. There are several interpolation methods, such as linear interpolation, Lagrange interpolation, Newton interpolation, and others. Each of them has its advantages and limitations depending on the specific task.

Spline approximation is a method that splits an interval into several segments and approximates a function using piecewise polynomial functions called splines [4]. Splines have smoothness and allow more accurate approximation of complex functions, taking into account the peculiarities of the data [5].

Least squares methods are used to find the best approximation of a function by minimizing the sum of squared deviations between the approximating function and the original data. This method allows you to take into account measurement errors and noise in the data, providing the best approximation in terms of least squares.

One of the important areas where approximation methods play a key role is numerical integration. Numerical integration is used to calculate the values of certain integrals when an analytical solution is not available or inefficient. Approximation methods, such as quadrature formulas and Monte Carlo methods, allow you to approximate the values of integrals and obtain results with a given accuracy.

One of the common methods of stabilization is the feedback method. It consists in adding feedback to the system of difference equations in order to suppress possible instabilities and improve its dynamic properties. Feedback can be implemented both linear and nonlinear, and its choice depends on the specific system and the required characteristics.

Another method of stabilization is the use of additional control signals. These signals can be specially designed to maintain the stability of the system and provide the desired characteristics. For example, to stabilize systems of difference equations with nonlinear elements, the adaptive control method can be used, which allows the system to independently adjust control parameters in accordance with changing conditions.

There are also stabilization methods based on the use of filters and compensators. Filters are used to suppress noise and interference in the system, as well as to improve signal quality. Compensators, in turn, are designed to compensate for some dynamic characteristics of the system, such as lag or uneven response.

An important aspect of stabilization methods is the mathematical analysis and study of the stability of systems of difference equations. This includes determining the stability conditions, constructing stable areas, and analyzing the influence of parameters on the stability of the system. This analysis allows you to optimize the stabilization process and choose the most appropriate methods and strategies.

**Problem statement.** To demonstrate the calculations in numerical integration methods, consider the following equation:

$$\int[a,\ b]f(x)dx \approx \frac{h}{2}\times\left[f(a)+2\Sigma[i=1,\ n-1]f(xi)+\ f(b)\right] \tag{1}$$

integration boundaries, f(x) is an integrable function, h is an integration step, n is the number of integration steps, xi is the values of x at each step.

For example, take the equation:

$$\int[0,1]\ x^2\ dx \tag{2}$$

Using the numerical integration method, we apply the rectangle formula:

$$\int[a,\ b]\ f(x)\ \approx\ h\times\sum[i=1,\ n]\ f(xi) \tag{3}$$

where $xi = a+\left(i-\frac{1}{2}\right)\times h$,

where i = 1, 2, ..., n.

In our case, a = 0, b = 1, f(x) = $x^2$. Let n = 4

For each subsequent integration step, in order to obtain a sequence of values of states and control actions during a given time interval, it should be noted that the accuracy and efficiency of integration depends on the chosen numerical integration method and integration step, as well as on the properties and features of the system of difference equations itself. If necessary, you can use more complex integration methods with a smaller step to improve the accuracy of the result.

Then the integration step $h=\frac{(b-a)}{n}=\frac{(1-0)}{4}=0,25$

Calculate the values of xi:

$$x1 = 0+\left(\frac{1}{2}\right)\times 0,25 = 0,125$$

$$x2 = 0+\left(\frac{3}{2}\right)\times 0.25 = 0,375$$

$$x3 = 0+\left(\frac{5}{2}\right)0,25 = 0,625$$

$$x4 = 0+\left(\frac{7}{2}\right)\times 0,25 = 0,875$$

Now substitute the values of xi into the function f(x) = $x^2$ and calculate:

$$f(x1)=(0.125)^2 = 0.015625$$

$$f(x2)=(0.375)^2 = 0.140625$$

$$f(x3)=(0.625)^2 = 0.390625$$

$$f(x4)=(0.875)^2 = 0.765625$$

Now we apply the rectangle formula to calculate the approximate value of the integral:

$$\int[0,1]x^2dx \approx 0,25\times\left[f(x1)+\ f(x2)+\ f(x3)+\ f(x4)\right]=$$
$$0,25\times\ [0,015625+0,140625+0,390625+0,765625]=0,328125$$

Thus, the approximate value of the integral $\int[0,1]\ x^2\ dx$ is 0 328125, obtained using the numerical integration method and the rectangle formula.

Continuing the complex calculation in the methods of stabilization of systems of difference equations, let's look at a more detailed example.

Suppose we have a system of difference equations of the following form:

$$x[k+1]\ =\ Ax[k]\ +\ Bu[k]\ y[k]\ =\ C\times x[k] \tag{4}$$

here x[k] is the vector of the system state at time k, u[k] is the vector of input signals at time k, y[k] is the vector of output signals at time k, A, B, C are the matrices of the system.

For simplicity of calculations, assume that the matrices have the following form:

$$A = \left[[0.8, 0], [0, 0.9]\right]$$

$$B = [[0.2], [0.1]]$$

$$C = \left[[1, 0]\right]$$

Our goal is to stabilize the system, that is, to place the poles inside a single circle.

To do this, we will use the feedback method with positive feedback and select the feedback matrix K.

The feedback is set as follows:

$$u[k] = -K \times x[k] \tag{5}$$

Substitute it back into the system:

$$x[k+1] = Ax[k] + B(-Kx[k]) = (A - BK) \times x[k] \tag{6}$$

Now we have a new system with a matrix $A' = A - B \times K$.

To stabilize the system, it is necessary to choose a suitable feedback matrix K so that all the eigenvalues of the matrix $A'$ are inside the unit circle.

One of the approaches to choosing the K matrix is to use the full control method. To do this, we can use the formula:

$$K = R \times \left(B^{T}\right) \times P \tag{7}$$

where R and P are matrices found from the solution of the algebraic Riccati equation:

$$A'^{(T)}PA' - P + Q - A'^{(T)}PBR^{(-1)}B^{T}PA' = 0 \tag{8}$$

where Q is the matrix denoting the weight of the error that we want to minimize.

Solving the equations according to this principle, we can always get the matrix K, which will ensure the stabilization of the system.

**Solving the problem**. The solution of systems of difference equations is of great importance in the field of automation and mechanics, where dynamic systems play a key role in modeling and controlling various processes. Difference equations are discrete analogs of differential equations and are widely used to describe the dynamics of systems that change their values at discrete points in time.

One of the main tasks in automation and mechanics is the development of effective methods for solving systems of difference equations. These methods allow analyzing and predicting the behavior of the system over time, optimizing its parameters, designing and implementing control algorithms.

An important aspect in choosing the method is to ensure the stability and accuracy of the solution of the system of difference equations, as well as taking into account the features of the system and its dynamics. We will find a solution to the system:

$$x_{k+1} = \begin{bmatrix} 0,8 & 1 \\ -0,15 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0,5 \end{bmatrix} g_k, x_0 = \begin{bmatrix} 0,5 \\ -1 \end{bmatrix}, \tag{9}$$

$$y_k = [21] x_k \tag{10}$$

where, $x_k$ and $y_k$ by the $z$ -transformation method, when $T = 0,8c$ and $g_k = kT$ .

First of all, only with the help of the method $z$ -transformations $g(z)$ of external influence $g_k = kT$ :

$$g(z) = \frac{0,8z}{(z-1)^2} \tag{11}$$

Next, we subject the z-transformation under non-zero initial conditions to obtain

$$zx(z) - zx_0 = \begin{bmatrix} 0,8 & 1 \\ -0,15 & 0 \end{bmatrix} x(z) + \begin{bmatrix} 1 \\ 0,5 \end{bmatrix} g(z)$$

From here, taking into account (11), we deduce

$$\left( zE - \begin{bmatrix} 0,8 & 1 \\ -0,15 & 0 \end{bmatrix} \right) x(z) = zx_0 + \begin{bmatrix} 1 \\ 0,5 \end{bmatrix} \frac{0,8z}{(z-1)^2} \tag{12}$$

To find $x(z)$ from this expression , we calculate first solve the matrix

$$A^{-1}(z) = \text{adj}\frac{A(z)}{\det A(z)} \text{ In our case}$$

$$A(z) = \begin{bmatrix} z-0,8 & -1 \\ 0,15 & z \end{bmatrix}, \ \text{adj}A(z) = \begin{bmatrix} z & 1 \\ -0,15 & z-0,8 \end{bmatrix}$$

$\det A(z) = z^2 - 0,8z + 0,15$ . The roots of equation $z^2 - 0,8z + 0,15 = 0$ are $z_1 = 0,3; z_2 = 0,5$. Therefore $\det A(z) = (z-0,3)(z-0,5)$ , and the matrix

$$A^{-1}(z) = \begin{bmatrix} z & 1 \\ -0,15 & z-0,8 \end{bmatrix} \frac{1}{(z-0,3)(z-0,5)}.$$

Multiplying equation (12) by matrix $A^{-1}(z)$ on the left, we get

$$x(z) = \frac{z}{(z-0,3)(z-0,5)} \begin{bmatrix} z & 1 \\ -0,15 & z-0,8 \end{bmatrix} \left\{ x_0 + \begin{bmatrix} 1 \\ 0,5 \end{bmatrix} \frac{0,8}{(z-1)^2} \right\}. \tag{13}$$

First, we calculate the component $x_{св}(k)$ due to the initial conditions. Her image

$$x_{св}(z) = \frac{z}{(z-0,3)(z-0,5)} \begin{bmatrix} z & 1 \\ -0,15 & z-0,8 \end{bmatrix} \begin{bmatrix} 0,5 \\ -1 \end{bmatrix} = \frac{z}{(z-0,3)(z-0,5)} \begin{bmatrix} 0,5z-1 \\ 0,725-z \end{bmatrix}. \tag{14}$$

In order to determine $x_{св}(k)$ from (14) using transformation tables, we decompose the ratio into the simplest fractions

$$\frac{Az+B}{(z-0,3)(z-0,5)} = \frac{D}{z-0,3} + \frac{C}{z-0,5}, \tag{15}$$

where $A$ and $B$ are the given coefficients available in (14), and $D$ and $C$ are unknown coefficients to be found. Applying our methodology, we get

$$D = \left.\frac{Az+B}{z-0,5}\right|_{z=0,3}, \quad C = \left.\frac{Az+B}{z-0,3}\right|_{z=0,5}. \tag{16}$$

Comparing (14) with (15) and applying formulas (16), we find:

$$x_{cв}(z) = z\begin{bmatrix} \dfrac{0,5z-1}{(z-0,3)(z-0,5)} \\ \dfrac{-z+0,725}{(z-0,3)(z-0,5)} \end{bmatrix} = \begin{bmatrix} \dfrac{4,25z}{z-0,3} - \dfrac{3,75z}{z-0,5} \\ \dfrac{-2,125z}{z-0,3} + \dfrac{1,125z}{z-0,5} \end{bmatrix}.$$

Next, using the z-transformation tables, we get $x_{cв}(k) = \begin{bmatrix} -3,75\cdot(0,5)^k + 4,25\cdot(0,3)^k \\ 1,125\cdot(0,5)^k - 2,125\cdot(0,3)^k \end{bmatrix}.$

For verification , let ‘s put $k=0$ here , then $x_{cв}(0) = \begin{bmatrix} 0,5 & -1 \end{bmatrix}^T$ . Since the forced component of the solution will be zero at $k=1$ (since there are no $\delta$ functions in the input effect), $x_{\text{св}}(0)$ should be equal to the given vector $x_0$. In our case, this condition is obviously fulfilled.

Let ‘s move on to the definition of the forced component $x_{\text{вын}}(k), z$ -the image of which is determined by the expression.

$$x_{\text{вын}}(z) = \begin{bmatrix} z & 1 \\ -0,15 & z-0,8 \end{bmatrix}\begin{bmatrix} 1 \\ 0,5 \end{bmatrix}\frac{0,8z}{(z-1)^2(z-0,3)(z-0,5)}$$

or

$$x_{\text{вын}}(z) = \begin{bmatrix} 0,8z+0,4 \\ 0,4z-0,44 \end{bmatrix}\frac{z}{(z-1)^2(z-0,3)(z-0,5)}. \tag{17}$$

As above, we decompose into the simplest fractions the corresponding ratios in (17), i.e.

$$\frac{Az+B}{(z-1)^2(z-0,3)(z-0,5)} = \frac{C}{(z-1)^2} + \frac{D}{z-1} + \frac{K}{z-0,3} + \frac{L}{z-0,5}.$$

Since there is a multiple pole $(z=1)$ here, it is impractical to apply formulas similar to (13). In this regard, we will find additional factors for each fraction and compose a system of algebraic equations of the following form:

$$\begin{bmatrix} 0,15 & -0,15 & -0,5 & -0,3 \\ -0,8 & 0,95 & 2 & 1,6 \\ 1 & -1,8 & -2,5 & -2,3 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} C \\ D \\ K \\ L \end{bmatrix} = \begin{bmatrix} B \\ A \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

This system is obtained by equating coefficients with the same degrees in the numerator of the right and left parts of equality (17), after bringing its right part to a common denominator. It is written by substituting the coefficients of additional multipliers for each term in the right part (17) into the corresponding columns of the matrix in the left part of the system (18).

**Conclusion.** One of the key elements of optimal control of variable systems is the definition of an objective function that reflects the required optimality criteria, such as minimizing costs, maximizing efficiency or achieving a certain level of productivity. Then, based on the system model and taking into account constraints, various optimization methods are used to find optimal control actions, and also related to issues of safety, reliability and stability [6]. It is important to take into account the possibility of failures, errors or external influences and develop management strategies that ensure the stability of the system, minimize potential risks and protect against undesirable situations [7].

Optimal control of variable systems in the dynamic spectrum is a complex and multifaceted field of research that plays a key role in automation and has wide application in various fields of human activity. Automation, as the science of automatic control and control of systems, strives to ensure the optimal operation of the system, taking into account the set goals, limitations and conditions.

The application of optimal control of variable systems in automation also faces a number of difficulties and challenges. For example, the need to take into account uncertainty and noise in the system, changing operating conditions, non-linearity and complexity of models. In addition, the computational complexity of optimal algorithms can be high, requiring efficient methods and high-performance computing systems [8].

It is necessary to delve into various optimization methods, adapt them to the specific requirements of specific systems, and develop new approaches, taking into account the development of technologies and the emergence of new challenges. This will make it possible to achieve more efficient and sustainable management of variable systems and apply them in various fields.

## REFERENCES

[1]. Vashchenko, G. V., Novikov E.A. *Parallel implementation of the explicit Euler method with control of the accuracy of calculations.* Journal of the Siberian Federal University. Series: Mathematics and Physics, 2011, Vol. 4, No. 1, pp. 70-76. – EDN NCBQKR.

[2]. Shornikov, Yu. V. *Stability analysis of explicit Runge -Kutta methods in the ISMA instrumental environment*. Problems of Informatics. 2009. – № 2(3). – Pp. 42-51.

[3]. Tsitouras, C. (2007). *Runge–Kutta interpolants for high precision computations*. Numerical Algorithms, 2007, 44, pp. 291-307.

[4]. Gorozhankin, S.A., Shitov, A.A., Savenkov N.V. *Methods for approximating the dependencies of several variables in the MS Excel and Mathcad software environment.* Scientific and Technical Bulletin of St. Petersburg State Polytechnic University. Computer science. Telecommunications. Management, 2016, № 3(247), pp. 35-47. – DOI 10.5862/JCSTCS.247.4. – EDN WZRGKT.

[5]. Medvedev, A.V., Gorodetskaya, O.Y. *Modeling of optimal development of the region in the system of economic relations of the Russian Federation.* Chronoeconomics, 2017, № 4(6), pp. 27-30.

[6]. Artemyev, V.S., Savostin S.D. *Analytics and methods of mathematical modeling of artificial intelligence in automation of control and testing.* Agrarian Science - 2022: Materials of the All-Russian Conference of Young Researchers, Moscow, November 22-24, 2022. – Moscow: Russian State Agrarian University - Moscow Agricultural Academy named after K.A. Timiryazev, 2022, pp. 340-343.

[7]. Artemyev, V.S., Savostin S.D. *Automated resource security management systems in digitalization issues*. Agrarian Science - 2022: Materials of the All-Russian Conference of Young Researchers, Moscow, November 22-24, 2022. Moscow: Russian State Agrarian University, Moscow Agricultural Academy named after K.A. Timiryazev, 2022. pp. 344-347

[8]. Artemyev, V.S., Savostin S.D. *Automation of methods and algorithms of information and analytical systems*. Vavilov readings - 2022: Collection of articles of the International scientific and practical Conference dedicated to the 135th anniversary of the birth of Academician N.I. Vavilov, Saratov, November 22-25, 2022. – Saratov: Limited Liability Company "Amirit", 2022. – pp. 639-642.

[9] Debela, H.G. and Duressa, G.F. *Exponentially fitted finite difference method for singularly perturbed delay differential equations with integral boundary condition.* Int. j. eng. appl. sci., 2019, 11, pp. 476–493.

[10] Debela, H.G. and Duressa, G.F. *Accelerated fitted operator finite difference method for singularly perturbed delay differential equations with non-local boundary condition.* J. Egyptian Math. Soc., 2020, 28, pp. 1–16.

[11] Debela, H.G. and Duressa, G.F. *Uniformly convergent numerical method for singularly perturbed convectiondiffusion type problems with nonlocal boundary condition.* Int J Numer Methods Fluids, 2020, 92, pp. 1914–1926.

[12] Kumar D. and Kumari P. *A parameter-uniform collocation scheme for singularly perturbed delay problems with integral boundary condition*. J. Appl. Math. Comput., 2020, 63, pp. 813–828.

[13] Sekar, E. and Tamilselvan, A. *Singularly perturbed delay differential equations of convectiondiffusion type with integral boundary condition.* J. Appl. Math. Comput., 2019, 9, pp. 701–722.

[14] Sekar, E. and Tamilselvan, A. *Third order singularly perturbed delay differential equation of reaction diffusion type with integral boundary condition*. J. Appl. Math. Comput. Mech., 2019, 18, pp. 99-110.

[15]. Gorodetskaya, O.Y., Gobareva, Ya. L., Medvedev, A.V. *Forecasting of financial markets using a convolutional neural network.* Problems of Economics and legal practice. 2021, Vol. 17, No. 3. pp. 65-72. – EDN NSUELT.

[16]. Sekar, E. Tamilselvan, A. *Finite difference scheme for third order singularly perturbed delay differential equation of convection diffusion type with integral boundary condition*. Journal of Applied Mathematics and Computing, 2019, №61, pp. 73–86.

[17]. Selvi, P. A. and Ramanujam, N. *A parameter uniform difference scheme for singularly perturbed parabolic delay differential equation with robin type boundary condition.* Applied Mathematics and Computation, 2017, №296, pp. 101–115.

[18]. Amiraliyev, G. M., Amiraliyeva, I. G., and Mustafa, K., *A numerical treatment for singularly perturbed differential equations with integral boundary condition*, Applied Mathematics and Computation, 2007, №185, pp. 574–582.

[19] Ansari, A.R., Bakr, S.A. and Shishkin, G.I. *A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations.* Journal of Computational and Applied Mathematics, 2007, №205, pp. 552–566.

**Received:** 21.10.2022
**Accepted:** 15.03.2023