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A NOVEL TWO-POLYNOMIALS CRITERIA FOR HIGHER-ORDER SYSTEMS STABILITY BOUNDARIES DETECTION AND CONTROL

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Abstract. There are many methods of identifying general stability of complex dynamic systems. Routh and Hurwitz's criterion is one of the earliest and commonly used analytical tools analysing stability of a dynamic system. However, it requires redundant calculation of all the elements of the Routh array to identify stability, even the low-order system. Therefore, it is not a simple method to identify, especially analytically, the stability boundaries for the coefficients of the characteristic equation due to tedious and lengthy derivation of all the Routh array elements. The proposed brand-new criterion or algorithm is an effective alternative and a universal technique to identify analytically the stability of up to sixth-order linear time-invariant dynamic system based on the set of unique for all possible system equations (2) and (3) that relate coefficients of the system characteristic polynomial at the stability boundaries by means of a single additional constant *k*. The expressions derived on this basis for a higher-order dynamic system can be used effectively to identify the boundaries of its stable behaviour spans. It defines the necessary and sufficient conditions for absolute stability of higher-order dynamic systems. It also allows the analysing of the system's precise marginal stability condition (whether stable or not) and the nature of the system roots at the stability boundaries, i.e. when they are relocated on imaginary $j\omega$-axis of s-plane. The criterion proposed by the authors, in contrast to Routh criteria, simplifies significantly the identification of maximum and minimum stability limits for any coefficient of the higher-order characteristic equation. The paper also presents the numerical analysis of stability boundaries for systems with order higher than six based on criteria (2) or (3). The derived stability boundary formulas (2) and (3) for the polynomial coefficients are successfully used for PID controller gains selection in close-loop control systems and this achievement does not have analogy in control theory.

Keywords: *higher-order dynamics, characteristic polynomial, stability boundaries, absolute and marginal stability.*

Introduction. The research on stability of higher-order systems was initiated by Edward Routh and Adolf Hurwitz long ago, their theory is being used now by control experts while analysing stability of dynamic systems and added to many books on control engineering [1-4]. It provides an effective tool for identifying stability condition dynamic system and roots of its system polynomial on the $j\omega$-axis of s-plane. However, it does not provide an effective method for identifying precise stability limits of higher-order system operation in analytical or numerical way by mathematically analysing the coefficients of the system characteristic polynomial. Deriving analytical expressions based on the Routh array is a very tedious and lengthy process. It becomes formidable task for systems with order higher than four. Besides, for special cases of all zeros in an array raw, the use of standard Routh procedure does not provide solution to the problem.

Some researchers have managed to solve specific system stability problems by using the Routh-Hurwitz criterion. In paper [5], the authors use the Hermite-Biehler theorem to derive Routh-Hurwitz criterion and managed to capture the system's unstable root counting. While performing stability analysis the Routh array may suffer some singularities. One example is when the first element of a row turns out to be zero. The solution to this case was discussed in some papers [5, 6, and 7] and textbooks [1-4]. Some researchers have used the ϵ-method to solve the stability problem for the special case when there are zero leftmost elements together with an all-zero row in the Routh

array [6]. A minor reconstruction of Routh's array is presented in [7] to solve a special case of leading array elements in the array becoming zero. In reconstructed array, locations of of a polynomial root are defined by means of considering first-column sign changes, similar to Routh's method, which eliminates the necessity of implementing the ϵ-approach.
The singularity in Routh array may also occur in case where all elements of a row become zero. In [8], the authors have presented a solution for the roots of a polynomial in the right-half of *s*-plane and on the *jω*-axis for the case when a few elements of a row in the Routh array become zero. They have used the continued fraction approach to solve the problem. When a system parameter is of the ϵ-order, the advantage of the ϵ-method of the Routh-Hurwitz criterion for the zero row was elaborated in [9]. In [10], authors have replaced zero row coefficients with the derivative of the polynomial corresponding to the row next to the zero-row to fill the row as an additional procedure and doing that they have succeeded in identifying the location of the symmetric roots of the polynomial on the right and left and/or on the *jω*-axis. [7].

Importantly, the Routh-Hurwitz criterion unable to determine the case of instability for the case of multiple roots on the *jω*-axis of the *s*-plane [2, 4, and 11]. Routh array does not provide solution for the number of *jω*-axis roots with multiplicity greater than unless solving it with the auxiliary polynomial. Moreover, Routh's array does not show sign change in the first column of the array for *some unstable systems* with repeated multiple roots on *jω*-axis even by applying the auxiliary polynomial procedure, i.e. indicating that there are no roots of the system polynomials in the right half *s*-plane [11]. In [10], the authors are managed to count the number of roots on *jω*-axis that are complex polynomials. The authors in [12] have investigated possible relation between the multiplicity of *jω*-axis poles and the numbers of zero rows in the Routh array. The main outcome was a prove that existence of more than one zero row in the Routh array is a source of instability of the system regardless of any sign change in the first column. In paper [13], authors have aimed for modelling and analysis of cyclic physical phenomenon and investigated harmonic oscillations of higher-order systems at the borders of stability regions. Stability boundary oscillations are used in many science and engineering applications [13]. The authors in [14, 15] conducted boundary locus analysis to achieve a stable control system design. The authors identified stability regions of controller coefficients based on a solution of characteristic polynomial equation in *s* domain for $s=j\omega$. In the research paper [13], the authors have identified the harmonic oscillation boundary of systems by mapping the roots of the characteristic polynomial to amplitude-angle $(M-\theta)$ plane and presenting roots of polynomial in the form of $\lambda = Me^{j\theta}$.

Another common method of *n*-th order systems stability studies is related to analysing numerical eigenvalues of of *n* state equations [16, 17]. However, it does not simplify the solution of the problem for *n*-th order system, the dimensions of a matrix of eigenvalues and matrix **A**, i.e (λ**I**-**A**), are of the same *n*-th order. Therefore, the level of complexity of stability problem solution is similar to analysing roots of the original *n*-th order system characteristic polynomial. In other words, it requires calculation numerically the roots λ of *n*-th order polynomial to verify stability of a given system and therefore analytical solution of the problem is not possible.

The literature review has shown that so far there is no any systematic and exact solution for stability problem of linear higher-order dynamic systems that is able to identify exact stability boundaries of system behaviour in terms of the coefficients of its characteristic polynomial and doing that is able thoroughly analyse and differentiate marginal stability or instability of systems at the boundary regions of stability. The importance of such theory could also contribute to closed-loop controllers design and selection of controller gains for real dynamic systems. The closed-loop controller gains are part of the system characteristic polynomial coefficients and, therefore, stability limits of the coefficients can be used, in turn, to identify stability limits for the gains. The method described in this paper aims to solve these problems. In addition, it can precisely define the number and types of conjugate roots on the *jω*-axis of the *s*-plane while dynamic system is at the stability boundary region and their influence on marginal stability or instability for some special cases of zero coefficients.

The new theory of stability was initially introduced in [18]. However, current paper in section II introduces a completely modified systematic and more simple approach for identifying stability of higher-order linear time-invariant dynamic systems with only two polynomials as an alternative to the renowned Routh-Hurwitz criterion and any other method. The discovered criterion and algorithms for system stability has no analogy to other criteria published so far in the field of stability control. The algorithms in this paper have been developed intuitively based on certain systematic relations of the coefficients of characteristic polynomial at the boundaries of stability and they are undoubtedly and successfully worked with all higher-order dynamic systems either with randomly selected coefficients of characteristic polynomials or randomly selected engineering applications with closed-loop controllers. The presented method is a simple and the only available procedure to identify the stability boundaries of the coefficients of a higher-order system characteristic polynomial. The difficulty of achieving the same objective by using the Routh has been discussed in section III for a sixth-order system. The presented algorithms are essential tools to identify marginal stability or instability of the systems for the case of multiple roots of the polynomials on $j\omega$-axis of the s-plane. Section III discusses in details and set the rules to identify stability boundaries for third-order, fourth-order, fifth-order and sixth-order dynamic systems in analytical form. The paper presents separate rules for absolute and marginal stability of systems when the multiple roots occur on the $j\omega$ -axis of the s-plane. Section IV also presents the numerical technique to identify stability boundaries for the dynamic systems with orders higher than six when it was difficult to derive analytical solution for the problem. For all the systems analysed in this paper, the boundary limits of lowest order dependent coefficients of characteristic polynomial a_0 as a function of other coefficients were determined. Section V demonstrate the use of the developed theory for defining of stability limits for the gains of closed loop controllers for various engineering systems with higher-order dynamic models.

Presentation of general stability criteria. In general the characteristic polynomial for higher-order dynamic system can be presented as follows:

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2}\, s^{n-2} + \cdots + a_1 s + a_0 \ = 0 \tag{1}$$

One of the conditions of possible stability is that all the coefficients of the polynomial must be positive real numbers [2.3]. However, positive values of the coefficients alone do not provide stability of the system. The current paper presents stability criteria of the higher-order systems system with all positive values coefficients as well as when some coefficients having zero values which leads to special cases of marginal stability or instability.

The general necessary stability criteria for any n-order dynamic system (where $n \geq 3$) can be uniquely expressed by the set of two non-linear algebraic equations (2) or (3) with introduction of an additional unknown variable k that couples both equations together. If the highest order of the system n is an *odd* number then two equations are presented, as follows:

$$\begin{aligned} a_n &= (a_{n-2} - (a_{n-4} - \cdots - (a_3 - a_1 k)k) \ldots)k \\ a_{n-1} &= (a_{n-3} - (a_{n-5} - \cdots - (a_2 - a_0 k)k) \ldots)k \end{aligned} \tag{2}$$

If the highest order of the system n is an *even* number then two equations are presented differently, as follows:

$$\begin{aligned} a_n &= (a_{n-2} - (a_{n-4} - \cdots - (a_2 - a_0 k)k) \ldots)k \\ a_{n-1} &= (a_{n-3} - (a_{n-5} - \cdots - (a_3 - a_1 k)k) \ldots)k \end{aligned} \tag{3}$$

It can be seen from (2) and (3) that unknown parameter k must be a real positive number to ensure that coefficients a_n and a_{n-1} are positive real numbers which is obvious stability condition for the system.

The *fundamental law of marginal or boundary stability* of any dynamic system with order $n \geq 3$ is stated as follows: "if equations (2) or (3) are satisfied and there exists solution of these equations with at least one common k as a positive real root, then all the coefficients in (1) are having stability boundary values and the system under consideration is in the state of marginal or boundary stability condition". At this state some of the roots of characteristic polynomial (1) form conjugate pairs and strictly located on the imaginary $j\omega$-axis of the s-plane. Therefore, (2) or (3) represent the *necessary and sufficient* criteria to define accurately stability boundary value for all the coefficient of dynamic system characteristic polynomial with order $n \geq 3$, provided algebraic equations (2) or (3) have at least one common positive real solutions for k. In other words, if conditions (2) or (3) satisfy, then the dynamic system is in the state of marginal stability or instability, i.e. it is exactly in between the stable and unstable zones of behaviour. The boundary values for the coefficients of n-th order system (1) can be obtained by mathematically excluding unknown k from both equations (2) or (3). The newly developed expressions (2) or (3) have no similarity to any stability criteria published so far in the literature. The relationship between the coefficients of the characteristic polynomial at the state of system stability boundary regions has been discovered intuitively but can be verified by any other method that describes stability boundary conditions for a dynamic system.

The lowest order of the system to be solved with (2) or (3) is n=3. For this 3rd order system (odd order), expression (2) can be expressed as follows:

$$a_3 = (a_1)k \quad \text{an} \quad a_2 = (a_0)k$$

Excluding k from both equations can lead to the stability boundary expression for the third order system, i.e. $a_2/a_0 = a_3/a_1$. This system is fully stable if the following inequality is satisfied:

$$\frac{a_2}{a_0} \geq \frac{a_3}{a_1} \tag{4}$$

The equal sign means a marginal stability of the system with *one pair of roots* symmetrically located on the $j\omega$-axis of the s-plane. The validity of (4) can be verified by using other methods such as Routh criteria and root-locus method.

Analytical solution for the stability boundaries for up to sixth-order systems.

a. *Stability criterion for the fourth-order dynamic system*

The system is presented by the following characteristic polynomial:

$$a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0 = 0 \tag{5}$$

Since the highest order of the system n=4, i.e. it is an even number, equations (3) were used to describe the boundary conditions for the polynomial (5).

$$a_4 = (a_2 - a_0 k)k \tag{6}$$
$$a_3 = (a_1)k \tag{7}$$

According to the newly developed fundamental law of boundary stability condition of any dynamic system with with order $n \geq 3$, values of k in (6) and (7) must be real positive numbers. Solving (6) and (7) for k leads to the following results:

$$k = \frac{\left(a_2 \pm \sqrt{a_2^2 - 4a_0 a_4}\right)}{2a_0}; \quad k = \frac{a_3}{a_1}$$

From these expressions part of the *necessary* boundary stability conditions for the system (5) can be derived. For *k* to be positive real number the following conditions must satisfy:

$$a_2^2 \geq 4a_0a_4;\ a_1 \neq 0;\ a_3 \neq 0 \tag{8}$$

Excluding *k* from both (6) and (7) and some algebra leads to the following single *necessary boundary or marginal stability* expression for the coefficients of (5):

$$a_0{a_3}^2 = (a_2a_3 - a_1a_{4)}a_1 \tag{9}$$

For a_0 to have positive non-zero value the following *necessary* boundary stability condition must be satisfied:

$$a_2a_3 > a_1a_4 \tag{10}$$

As a conclusion, it can be stated that (9) presents the *necessary and sufficient boundary or marginal stability* expression for the coefficients of (5) provided (8) and (9) are fully satisfied. The analysis shows that when the coefficients reach their stability boundary values, the system becomes *marginally stable* with *two (i.e. one pair of conjugate complex numbers) out of four roots located symmetrically on jω*-axis *of the s-plane*.

For comparison and *prove of validity* for (9), the expressions of the Routh array elements (two columns with five rows) for the fourth-order system were presented. The final equation in the first column of the Routh array can be presented as follows:

$$a_1 - \frac{a_3a_0}{A} = 0 \tag{11}$$

Substituting A=$(a_2a_3 - a_1a_4)/a_3$ into (11), the same expression (9) can be derived. Therefore, Routh method solution serves as an *'elementary' prove* of validity for the proposed stability criteria (3).

The expression (9) can be easily used to define the boundary values of any coefficient of the characteristic equation (5). For example, for this system to be *absolutely stable*, a_0 must be within the range in between its minimum and maximum boundary values, as it was defined from (9):

$$0 < a_0 < \frac{(a_2a_3 - a_1a_{4)}a_1}{{a_3}^2} \tag{12}$$

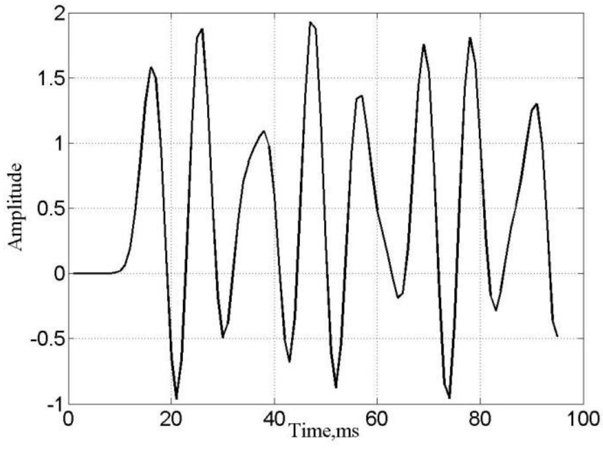

Figure 1 Marginally stable system

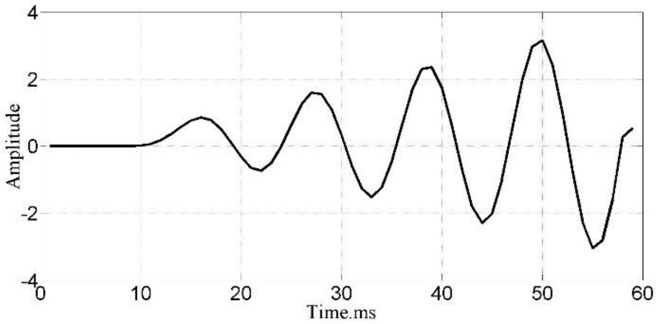

Figure 2 Marginally unstable system

In accordance to (9), in the case $a_1 = a_3 = 0$, i.e. when (9) yields 0=0, the following *special case* of marginal stability or marginal instability when *all four roots of polynomial are located on jω -axis* of the *s*-plane has been derived. If $a_1 = a_3 = 0$, the system becomes marginally stable and its characteristic polynomial will have *all four roots (in this case two different pairs of conjugate complex numbers) symmetrically located on jω-axis of the s-plane if* $a_2^2 > 4a_0a_4$, or if still $a_1 = a_3 = 0$, the system becomes marginally unstable and its characteristic polynomial will *have all four roots* (in *this case two repeated pairs of conjugate complex numbers) symmetrically located on jω*-axis *of the s-plane if* $a_2^2 = 4a_0a_4$. This special case marginal stability is a unique case when the same condition of coefficients $a_1 = a_3 = 0$ lead to two different behaviour of the 4th order system and different type of roots of the polynomial (5) depending of the values of its remaining coefficients.

The example of a system polynomial that has *marginal stability* is $s^4 + 3s^2 + 2 = 0$ and Fig. 1 shows response of this system to the unity input. The example of a system polynomial that has *marginal instability* is $s^4 + 4s^2 + 4 = 0$ and Fig. 2 shows response of this system to the unity input. The difference in both system responses with the same condition for the coefficients $a_1 = a_3 = 0$ can be seen clearly in Fig. 1 and Fig. 2. This unique condition of marginal stability or instability was not identified by using the Routh method of stability analysis.

b. Stability criterion for the fifth-order dynamic system

The system is presented by the following characteristic polynomial:

$$a_5s^5 + a_4s^4 + a_3\,s^3 + a_2\,s^2 + a_1s + a_0 \; = 0 \tag{13}$$

Solving the boundary problem for a system polynomial (10) coefficients can be done by taking *n*=5 (odd number) in (2). It leads to the following two non-linear equations:

$$a_5 = (a_3 - a_1k)k \tag{14}$$

$$a_4 = (a_2 - a_0k)k \tag{15}$$

According to the newly developed fundamental law of boundary stability condition of any dynamic system with with order n ≥3, values of *k* in (14) and (15) must be real positive numbers. Solving (14) and (7) for *k* leads to the following results:

$$k = \frac{\left(a_3 \pm \sqrt{a_3^2 - 4a_1a_5}\right)}{2a_1}; \;\; k = \frac{\left(a_2 \pm \sqrt{a_2^2 - 4a_0a_4}\right)}{2a_0}$$

From these expressions the *necessary and sufficient* boundary stability conditions for the system (13) can be derived. For *k* to be positive real number the following conditions for all six coefficients must satisfy:

$$a_3^2 \geq 4a_1a_5; \; a_2^2 \geq 4a_0a_4 \tag{16}$$

Excluding *k* from both (14) and (15) and some algebra leads to the following single *necessary boundary or marginal stability* expression for the coefficients of (13):

$$(a_1a_4 - a_0a_5)^2 = (a_1a_2 - a_0a_3)(a_3a_4 - a_2a_5) \tag{17}$$

Expression (17) requires the following additional conditions to be satisfied:

$$a_1a_2 > a_0a_3 \; and \; a_3a_4 > a_2a_5 \tag{18}$$

As a conclusion, it can be stated that (17) presents the *necessary and sufficient boundary or marginal stability* expression for the coefficients of (13) provided conditions (16) and (18) are fully satisfied. The analysis shows that when the coefficients reach their stability boundary values, the system becomes *marginally stable* with *two (i.e. one pair of conjugate complex numbers) out of five roots located symmetrically on jω*-axis *of the s-plane and with one of the roots (negative real number) located on the real axis of the s-plane*. In this case, the expressions (14) and (15) have only *one common* real positive root *k*.

The expression (17) can be easily used to define the boundary values of any coefficient of the characteristic equation (13). For example, for this system to be *absolutely stable*, a_0 must be within the range in between its minimum and maximum boundary values, as it was defined from (13):

$$a_0^{min} = \frac{(a_2a_5 - a_3a_4)\left(a_3 + \sqrt{a_3^2 - 4a_1a_5}\right) + 2a_1a_4a_5}{2a_5{}^2} \tag{19}$$

$$a_0^{max} = \frac{(a_2a_5 - a_3a_4)\left(a_3 - \sqrt{a_3^2 - 4a_1a_5}\right) + 2a_1a_4a_5}{2a_5{}^2} \tag{20}$$

If equation (19) yields a negative value, then a_0^{min} should be assigned a zero-boundary value instead.

In accordance to (17), in the case of $a_0a_5 = a_1a_4$, $a_0a_3 = a_1a_2$, *and* $a_2a_5 = a_3a_4$, i.e. when (17) yields 0=0, the following *special case* of marginal stability or marginal instability when *four roots of polynomial out of five are located on jω -axis* of the *s*-plane has been derived. If $a_0a_5 = a_1a_4$, $a_0a_3 = a_1a_2$, $a_2a_5 = a_3a_4$ and $a_3^2 > 4a_1a_5$, $a_2^2 > 4a_0a_4$, the system becomes <u>marginally stable</u> and the systems characteristic polynomial will have *four roots (two pairs of different conjugate complex numbers) located symmetrically on jω-axis of the s-plane and one root (negative real number) located on the real axis of the s-plane*. Subsequently, if $a_0a_5 = a_1a_4$, $a_0a_3 = a_1a_2$, $a_2a_5 = a_3a_4$ and $a_3^2 = 4a_1a_5$, $a_2^2 = 4a_0a_4$, the system becomes <u>marginally</u> *<u>unstable</u> but with four repeated roots (two pairs of same conjugate complex numbers) located symmetrically on jω-axis of s-plane and one root (negative real number) located on the real axis of the s-plane*. Should be notes that if conditions $a_0a_5 = a_1a_4$ *and* $a_0a_3 = a_1a_2$ are satisfied then condition $a_2a_5 = a_3a_4$ is satisfied by default as well. In both cases, the expressions (14) and (15) have *two common* real positive roots *k*.

The example of a system polynomial that has *marginal stability* is $4s^5 + s^4 + 8s^3 + 2s^2 + s + 0.25 = 0$ and Fig. 3 shows response of the system to the unity input. The example of a system polynomial that has *marginal instability* is $s^5 + s^4 + 8s^3 + 8s^2 + 16s + 16 = 0$ and Fig. 4 shows response of the system to the unity input. The difference in both system responses with the same conditions for the coefficient values $a_0a_5 = a_1a_4,\ a_0a_3 = a_1a_2,\ a_2a_5 = a_3a_4$ can be seen clearly in Fig. 3 and Fig. 4. This unique condition of marginal stability or instability was not identified by using the Routh method of stability analysis.

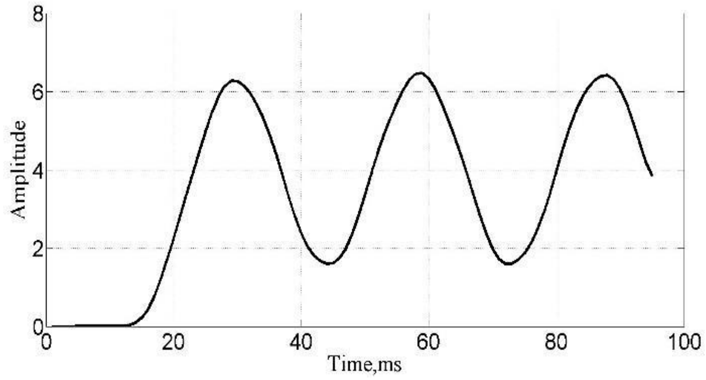

Figure 3 Marginally stable system (Rule 6)

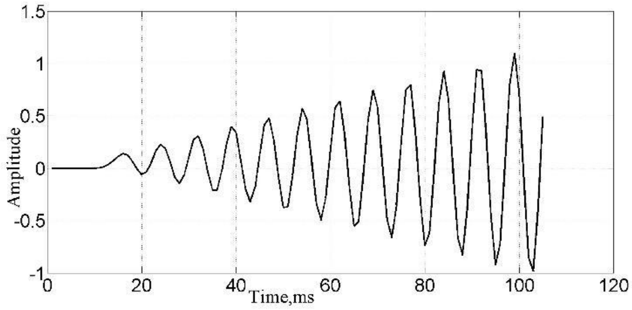

Figure 4. Unstable system (Rule 6)

c. Stability criterion for the sixth-order dynamic system

The sixth-order system is presented by the following characteristic polynomial:

$$a_6s^6 + a_5s^5 + a_4s^4 + a_3\,s^3 + a_2\,s^2 + a_1s + a_0\ = 0 \tag{21}$$

Solving the boundary problem for the system polynomial (21) coefficients can be done by taking *n*=6 (even number) in (3). It leads to the following set of two non-linear equations:

$$a_6 = (a_4 - (a_2 - a_0k)k)k \tag{22}$$
$$a_5 = (a_3 - a_1k)k \tag{23}$$

By excluding k from the expressions (22) and (23) the necessary marginal stability criteria for the system (21) coefficients can be analytically derived. Th resulting equation is expressed as follows:

$$Aa_6^2 - Ba_6 + C = 0, \tag{24}$$

where:

$$A = a_1^3,$$
$$B = (a_1a_2 - a_0a_3)(3a_1a_5 - a_3^2) + a_1^2(a_3a_4 - a_2a_5),$$
$$C = -a_5[(a_3a_4 - a_2a_5)(a_1a_2 - a_0a_3) - (a_1a_4 - a_0a_5)^2].$$

The positive solution for a_6 in equation (24) exists, if the following condition satisfies:

$$B^2 - 4AC > 0 \quad (25)$$

As a conclusion, it can be stated that (24) presents the *necessary and sufficient boundary or marginal stability* expression for the coefficients of (21) provided conditions (16), (18), and (25) are fully satisfied. The analysis shows that when the coefficients reach their stability boundary values, the system becomes *marginally stable* with *two (i.e. one pair of conjugate complex numbers) out of five roots located symmetrically on jω*-axis *of the s-plane and with one of the roots (negative real number) located on the real axis of the s-plane.* In this case, the expressions (22) and (23) have only *one common* real positive root k. The expression (24) can be easily used to define the boundary values of any coefficient of the characteristic equation (21).

In special case of marginal stability, when $B^2 = 4AC,\ a_6,$ has only one value and the sixth-order system (21) becomes marginally stable with four (*two pairs of different conjugate complex numbers)* *roots symmetrically located on jω-axis of the s-plane and two roots (one pair of conjugate complex numbers) located symmetrically on left half of the s-plane.* In this special case the expressions (22) and (23) have *two common* real positive roots k. The example of such system polynomial is shown below:

$$s^6 + 2s^5 + 4s^4 + 3s^3 + 4s^2 + 0.5s + 0.625 = 0$$

Fig. 5 shows the response of this system to the unity input. The validity of the Rules 7, 8, 9 have been verified by solving a numerical example of the systems with randomly selected polynomial coefficients.

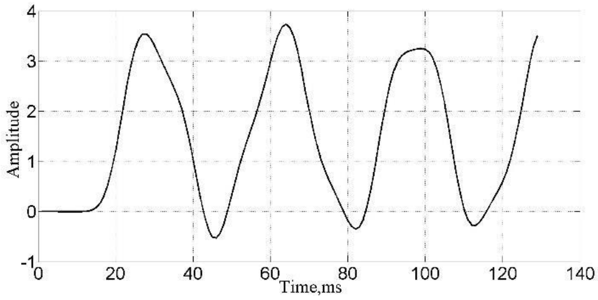

Figure 5 Marginally stable system

The attempt to derive stability boundary equation for the 6th order system by using Routh array (three columns and seven rows) leads to the following preliminary formulas:

$$ED - Ca_0 = 0$$
$$E = (BC - AD)/C$$
$$D = (Aa_1 - a_5a_0)/A$$
$$C = (Aa_3 - Ba_5)/A$$
$$B = (a_5a_2 - a_6a_1)/a_5$$
$$A = (a_5a_4 - a_6a_3)/a_5$$

Substituting these expressions one into another *does not* guarantee that the final expression for a_6 can be presented in the form of a quadratic equation and this demonstrates the deficiency and complexity of the Routh approach for the sixth and higher orders system stability analysis.

Numerical analysis of stability boundaries for the systems with orders higher than six. The analytical solution of the boundary problem (in form of radicals for k) for dynamic systems with any order is possible by systematically excluding k from equations (2) or (3) and finally reaching one polynomial equation with higher order for the coefficients of the characteristic equation (1). However, it becomes a tedious process for the systems with orders $n \geq 7$. In general, the system of any order is marginally stable if equations (2) or (3) satisfy the *fundamental law of marginal or boundary stability,* that is if they have at least one *common positive* roots k and have a pair or pairs of conjugate roots on *jω-axis of the s-plane*.

This low has been applied numerically to an eleventh order (n=11) system to prove validity of the law. The system with order n=11 is presented by the following characteristic polynomial:

$$a_{11}s^{11} + a_{10}s^{10} + a_9s^9 + a_8s^8 + a_7s^7 + a_6s^6 + a_5s^5 + \\ +a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 = 0 \qquad (29)$$

The stability boundary equations (2) for this system n=11 (odd number) can be presented in the following form:

$$a_{11} = (a_9 - (a_7 - (a_5 - (a_3 - a_1k)k)k)k)k \qquad (30)$$

$$a_{10} = (a_8 - (a_6 - (a_4 - (a_2 - a_0k)k)k)k)k \qquad (31)$$

Alternatively, (30) and (31) can be rewritten in the following forms with alternating signs for the terms:

$$a_{11} = a_9k - a_7k^2 + a_5k^3 - a_3k^4 + a_1k^5 \qquad (32)$$

$$a_{10} = a_8k - a_6k^2 + a_4k^3 - a_2k^4 + a_0k^5 \qquad (33)$$

In order to have a marginal stability condition, the roots of both fifth order polynomials (32) and (33) must have at least one real positive root in common. A randomly selected example of such equation (32) that has real positive roots is as follows:

$$0.3 = 2.7k - 7.2k^2 + 6.6k^3 - 2.4k^4 + 0.3k^5 \qquad (34)$$

Five roots of this equation are, in fact, real positive numbers:

$$k = [3.247, \mathbf{2.618}, 1.555, 0.382, 0.1981] \qquad (35)$$

In order for equation (33) to have at least one root to be the same with any root of equation (34), each of the k roots (35) is inserted into equation (33) with randomly selected coefficients except a_0. A random example of equation (33) can be presented as follows:

$$0.1 = 0.6k - 0.9k^2 + 0.5k^3 - 0.1k^4 + a_0k^5 \qquad (36)$$

Calculation of a_0 for each k root from the list (35) yields the following result:

$$a_0 = [0.004544, 0.0034, 0.0053, -2.9034, 41.7902] \qquad (37)$$

Then the polynomial (29) with all the preselected coefficients is presented as follows:

$$0.3s^{11} + 0.1s^{10} + 2.7s^9 + 0.6s^8 + 7.2s^7 + 0.9s^6 + \\ +6.6s^5 + 0.5s^4 + 2.4s^3 + 0.1s^2 + 0.3s + a_0 = 0 \qquad (38)$$

Solution of equation (38) for five possible a_0 from (37) shows that only two of them present actual stability boundaries for the polynomial (38), namely a_0^{min} = 0.003444 and a_0^{max} =0.004544. These two boundary values of a_0 in polynomial (38) lead to one pair of conjugate roots on the *jω-axis* of the s-plane, namely 0 ± 0.6180i and 0 ± 0.5550i. The system turns to be fully stable in between these two boundary values of a_0. The roots of boundary polynomial (36) at the boundary values of a_0 yield real positive values. For example, for a_0 = 0.004544 the roots of polynomial (36) yield k = [3.267, 2.6180, 1.8658, 1.0425, 0.2451]. Therefore, two boundary polynomials (32) and (33) have one common positive real root k=2.6180 as it was stated by the *fundamental law of marginal or boundary stability*.

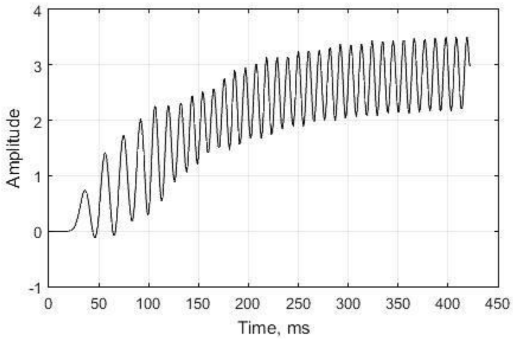

Figure 6 Marginally stable system

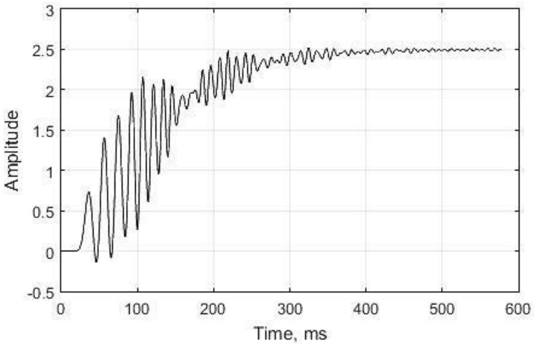

Figure 7. System with absolute stability

Fig. 6 shows the response of the system with characteristic polynomial (38) having value of a_0 =0.00344 to step input 0.01. From Fig. 6 it can be seen that the system is in the state of marginal stability. Fig. 7 shows the response of the system to step input 0.01 for a_0 selected in between its boundary values [0.003444, 0.004544]. From Fig. 7 it can be seen that the system is in the state of absolute stability.

Stability range for the closed-loop control systems. The *s*-domain transfer function for the closed-loop control system can be expressed as follows:

$$\frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)H(s)} \tag{39}$$

In (39), *R(s)* is the input signal, *Y(s)* is the output signal, *H(s)* is the feedback signal, *G(s)* is the plant model (system under observation), and *K(s)* is the controller model.

Expressions (2) or (3) can be successfully applied to identify stability ranges for the gains of closed-loop control system (39).

***a.** Case of single gain controller design*

The stability analysis of a system with single gain controller can be demonstrated on the *model of hard disk drive* with the *lead compensator*. The plant model of the hard disk drive system can be expressed as follows [19]:

$$G(s) = A/B \text{ , where} \tag{40}$$

$$A = n_4s^4 + n_3s^3 + n_2s^2 + n_1s + n_0,$$
$$B = d_{10}s^{10} + d_9s^9 + d_8s^8 + \cdots + d_4s^4 + d_3s^3 + d_2s^2,$$

where:

$$n_4 = 1.197 \cdot 10^{26},\ n_3 = 2.12 \cdot 10^{29}, n_2 = 5.826 \cdot 10^{34},$$
$$n_1 = 4.366 \cdot 10^{37},\ n_0 = 6.189 \cdot 10^{42}, d_{10} = 1,\ d_9 = 5336,$$
$$d_8 = 4.124 \cdot 10^{9},\ d_7 = 1.302 \cdot 10^{13},\ d_6 = 4.216 \cdot 10^{18},$$
$$d_5 = 6.72 \cdot 10^{21},\ d_4 = 1.198 \cdot 10^{27},\ d_3 = 7.496 \cdot 10^{29},$$
$$d_2 = 9.668 \cdot 10^{34}.$$

The *lead compensator* with a proportional gain k_p can be presented as follows:

$$K(s) = k_p(4s + 2)/(s + 2) \tag{41}$$

Substituting (40), (41) into (39) and assuming $H(s) = 1$, yields the following close-loop system characteristic polynomial (29), where:

$$a_{11} = d_{10}, a_{10} = d_9 + 2d_{10},\ a_9 = d_8 + 2d_9,$$
$$a_8 = d_7 + 2d_8, a_7 = d_6 + 12d_7, a_6 = d_5 + 2d_6,$$
$$a_5 = d_4 + 2d_5 + 4k_p n_4, a_4 = d_3 + 2d_4 + 2k_p(2n_3 + n_4),$$
$$a_3 = d_2 + 2d_3 + 2k_p(2n_2 + n_3),$$
$$a_2 = 2d_2 + 2k_p(2n_1 + n_2), a_1 = 2k_p(2n_0 + n_1),$$
$$a_0 = 2k_p n_0.$$

For the eleventh order characteristic polynomial (29), two stability boundary polynomials can be presented as (32) and (33). By substituting all the coefficients into (32) and (33) and dividing (32) by (33), proportional gain k_p can be excluded from the resulting single algebraic 6th order stability boundary equation with variable *k* as follows:

$$p_6k^6 + p_5k^5 + p_4k^4 + p_3k^3 + p_2k^2 + p_1k + p_0, \tag{42}$$

where:

$$p_6 = -dd_0nn_3 + dd_1nn_2 - dd_2nn_1 + dd_3nn_0,$$
$$p_5 = dd_0nn_5 - dd_1nn_4 + dd_2nn_3 - dd_3nn_2 + dd_4nn_1 - dd_5nn_{0,}$$
$$p_4 = -dd_2nn_5 + dd_3nn_4 - dd_4nn_3 + dd_5nn_2 - dd_6nn_1 + dd_7nn_0,$$
$$p_3 = dd_4nn_5 - dd_5nn_4 + dd_6nn_3 - dd_7nn_2 + dd_8nn_1 - dd_9nn_{0,}$$
$$p_2 = -dd_6nn_5 + dd_7nn_4 - dd_8nn_3 + dd_9nn_2 - dd_{10}nn_1 + dd_{11}nn_0,$$
$$p_1 = dd_8nn_5 - dd_9nn_4 + dd_{10}nn_3 - dd_{11}nn_2,$$
$$p_0 = -dd_{10}nn_5 + dd_{11},$$
$$dd_{11} = d_{10},\ dd_{10} = d_9 + 2d_{10},\ dd_9 = d_8 + 2d_9,$$
$$dd_8 = d_7 + 2d_8,\ dd_7 = d_6 + 2d_7,\ dd_6 = d_5 + 2d_6,$$
$$dd_5 = d_4 + 2d_5,\ dd_4 = d_3 + 2d_4,\ dd_3 = d_2 + 2d_3,$$
$$dd_2 = d_1 + 2d_2,\ dd_1 = d_0 + 2d_1,\ dd_0 = 2d_0,$$
$$nn_5 = 2n_4, nn_4 = 2n_3 + n_4, nn_3 = 2n_2 + n_3,$$
$$nn_2 = 2n_1 + n_2, nn_1 = 2n_0 + n_1, nn_0 = n_0.$$

Solution of equation (42) yields four real and two complex *k* roots. In accordance to Rule 10, only real roots could be considered for the marginal stability of the closed-loop system. Four real roots are 0.4912*10^{-6}, 0.0139*10^{-6}, 0.0077*10^{-6}, 0.0006*10^{-6}.

Value of $k_p$at the state of marginal stability can be calculated from (32) and presented as follows:

$$k_p = C/D, \text{ where} \tag{43}$$
$$C = dd_{11} - dd_9k + dd_7k^2 - dd_5k^3 + dd_3k^4 - dd_1k^5,$$
$$D = nn_5k^3 - nn_3k^4 + nn_1k^5.$$

Substituting four real roots of (42) into (43) yield three positive and one negative values of k_p. Negative value leads to instability of the system because the coefficient a_0 of the system is directly proportional k_p, i.e. $a_0 = 2k_p n_0$, and cannot be negative. As a result, the minimum stability limit for the k_p is zero, i.e. $k_{min} = 0$. The remaining three calculated positive value for k_p are 0.0079, 0.2119, 0.1726. Solving for the roots of characteristic polynomial (29) for these three values of k_p yields a pair of roots located on the imaginary axis of s-plane $\pm j0.1427 * 10^4, \pm j0.8488 * 10^4, \pm j1.1411 * 10^4$, respectively. The analysis of all solutions shows that only one gain value k_{pmax} =0.0079 corresponds to the marginal stability condition of the closed-loop system where all the roots located at the left half of s-plane. Three pairs of roots located on imaginary axis of s-plane $[\pm j0.1427, \pm j0.8488, \pm j1.1411] * 10^4$ can be verified by plotting root locus graphs and it is shown in Fig. 8.

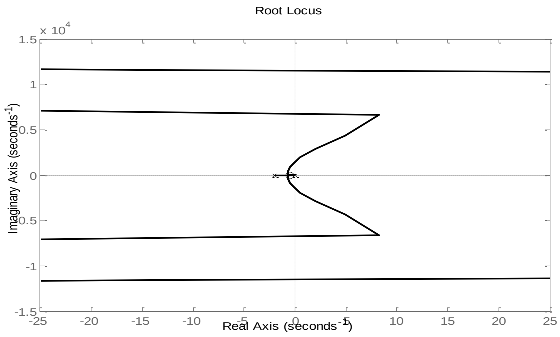

Figure 8. System with absolute stability

b. Case of multiple gain controller design

The advantage of applying expressions (2) and (3) for stability analysis of higher-order closed-loop dynamic systems can be demonstrated for the case of applying *multiple gain* controllers to the system. The criteria (2) and (3) was tested on the example of model of two-inertia system with PD controller. The plant model of such two-inertia system can be expressed as follows [20]:

$$G(s) = n_0/(d_4 s^4 + d_3 s^3 + d_2 s^2 + d_1 s + d_0), \text{ where: } \quad (44)$$
$$n_0 = 0.0625, d_4 = 1, d_3 = 2, d_2 = 1.5, d_1 = 0.5,$$
$$d_0 = 0.0625.$$

Substituting (44), $K(s) = k_p + sk_d$ into (39) and assuming $H(s) = 1$, yields the following fourth-order characteristic polynomial of the close-loop system:

$d_4 s^4 + d_3 s^3 + d_2 s^2 + (d_1 + n_0 k_d)s + \left(d_0 + n_0 k_p\right) = 0$ (45)

The two stability boundary polynomials (3) for the characteristic polynomial (45) can be presented as follows:

$$d_4 = k d_2 - k^2\left(d_0 + n_0 k_p\right), \quad (46)$$
$$d_3 = k(d_1 + n_0 k_d). \quad (47)$$

By dividing (46) by (47), the following expression for k_d can be derived:

$$k_d = [d_2 d_3 - d_3\left(d_0 - n_0 k_p\right)k - d_1 d_4]/n_0 d_4 \quad (48)$$

Substituting (48) into (47) yields the following quadratic equation:

$$\left(d_0 d_3 + d_3 n_0 k_p\right)k^2 - d_2 d_3 k + d_3 d_4 = 0. \quad (49)$$

Solution of (49) can be presented as follows:

$$k = [d_2 \pm \sqrt{d_2^2 - 4d_4(d_0 + n_0k_p)}]/2(d_0 + n_0k_p) \quad (50)$$

The stability boundary is achieved when the expression under square root is equal zero and solution of (50) yields a single positive answer for k (Rule 10). As a result, at the stability boundary condition for the system the expression for maximum limit of k_p can be derived from (50) as follows:

$$k_p^{max} = (d_2^2 - 4d_0d_4)/(4d_4n_0) \quad (51)$$

The minimum limit of k_p can be obtained from the condition that for a stable system all the coefficients of characteristic polynomial (45) must be positive. Therefore, the coefficient $d_0 + n_0k_p$ must have a positive value and the minimum value for k_p can be calculated as follows:

$$k_p^{min} = -d_0/n_0 \quad (52)$$

In order to provide an absolute stability of the closed-loop system, the following condition for k_p must be provided:

$$k_p^{min} < k_p < k_p^{max} \quad (53)$$

For any value of k_p within the limits (53), two values for ss are be calculated from (50) and subsequently two corresponding limit values for k_d can be calculated from (48). An additional condition for the system stability is that the minimum limit for k_d must be more than one calculated from the corresponding coefficient of the system characteristic polynomial, i.e.

$$k_d^{min} > -d_1/n_0. \quad (54)$$

Using all the stability conditions (54), (53), (52), (51), (50), and (48), the following graph of function $k_d = f(k_p)$ for the boundary values can be obtained, as shown in Fig. 9. For all the boundary values of the system gains, the solution of the characteristic polynomial (45) yields one pair of conjugate roots at the imaginary axis of s-plane, i.e. the system is at the condition of marginal stability.

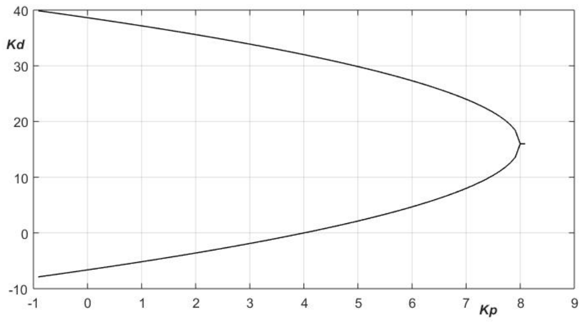

Figure 9. Stability boundary curves for $k_d = f(k_p)$

Fig. 9 shows the region of absolute stability of the system that lies in between upper and lower lines of the graph. The highest range of stability is at $k_p^{min} = -1$, where -8 $< k_d < 40$. At $k_p^{max} = 8$, the stability region is reduced to a single value k_d =16.

In case of applying PID controller $K(s) = k_p + sk_d + k_i/s$ to the model of two-inertia system [19], the following fifth order characteristic equation can be obtained:

$$d_4 s^5 + d_3 s^4 + d_2 s^3 + (d_1 + n_0 k_d) s^2 + \left(d_0 + n_0 k_p\right) s + n_0 k_i = 0 \tag{55}$$

Two stability boundary polynomials (2) for the characteristic polynomial (55) can be presented as follows:

$$d_4 = k d_2 - k^2 \left(d_0 + n_0 k_p\right), \tag{56}$$

$$d_3 = k(d_1 + n_0 k_d) - k^2 n_0 k_i \tag{57}$$

By dividing (56) by (67), the following formula for k_p can be obtained:

$$k_p = (d_2 d_3 - d_1 d_4 - n_0 d_4 k_d)/(n_0 d_3 k) - \\ -(d_0 d_3 - n_0 d_4 k_i)/(n_0 d_3) \tag{58}$$

Substituting (58) into (56) yields the following quadratic equation:

$$(n_0 k_i) k^2 - (d_1 + n_0 k_d) k + d_3 = 0 \tag{59}$$

Solution of (59) can be presented as follows:

$$k = \left[d_1 + n_0 k_d \pm \sqrt{(d_1 + n_0 k_d)^2 - 4 d_3 n_0 k_i}\right] / (2 n_0 k_i) \tag{60}$$

The stability boundary is achieved when the expression of square root in (60) is equal zero and solution of equation (60) yields a single positive answer for k (Rule 10). This condition yields the following boundary equation for k_d:

$$(n_0^2) k_d^2 + (2 n_0 d_1) k_d + d_1^2 - 4 n_0 d_3 k_i = 0 \tag{61}$$

The solution of (61) yields the boundary equation for k_d as follows:

$$k_d = -d_1 \pm 2\sqrt{d_3 n_0 k_i} \tag{62}$$

A stability boundary is achieved when the expression of square root in (62) is equal to zero, i.e. when k_i=0. Therefore, for an absolute stability of the closed-loop system, the following condition must be satisfied:

$$k_i > 0 \tag{63}$$

For any value $k_i > 0$, formula (62) yields two limiting values for k_d. The additional condition for the system stability is that the minimum limit for k_d must be more than one calculated from the corresponding coefficient of the system characteristic polynomial, i.e.

$$k_d^{min} > -d_1/n_0. \tag{64}$$

By substituting the two limiting values of k_d into (60) and subsequently into (58), the remaining two limiting values for k_p can be obtained. An additional condition for the system stability is that the minimum limit for k_p must be more than one calculated from the corresponding coefficient of the system characteristic polynomial, i.e.

$$k_p^{min} > -d_0/n_0. \tag{65}$$

Using all the stability conditions (65), (64), (63), (62), (60), and (58), the following 3D graph of function $k_p = f(k_d, k_i)$ for the boundary lines of k_p, k_d gains versus few values of k_i is shown in Fig. 10. The absolute stability of the system is confined within the space outlined by the limiting values of three gains.

Fig.11 shows only 2D view of the lines shown in Fig. 10. The maximum values for k_p, k_d, k_i gains are defined by the terminal condition when $k_p^{min} = k_p^{max}$ for raising in steps values of k_i (63) and is calculated on MATLAB software. Increasing k_i reduces that stability range of the system, i.e. stability ranges for other two gains. For all the boundary values of the system gains the solution of the characteristic polynomial (55) yields one pair of conjugate roots at the imaginary axis of s-plane, i.e. the system is at the condition of marginal stability. An exception is for the points where $k_p^{min} = k_p^{max}$. Fig. 12 shows a 2D graph of $k_p = f(k_d)$ for a single value k_i=0.

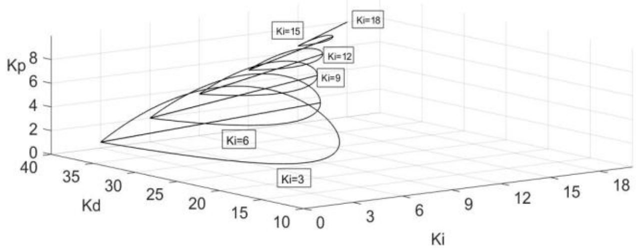

Figure 10. 3D Stability boundary curves for $k_p = f(k_d, k_i)$

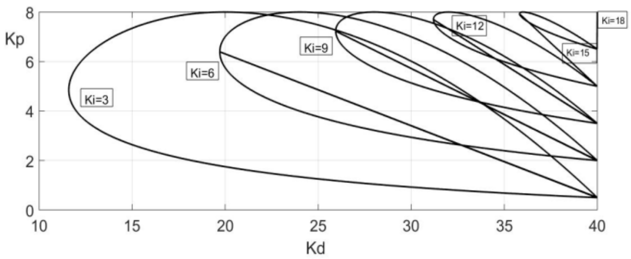

Figure 11. 2D Stability boundary curves for $k_p = f(k_d, k_i)$

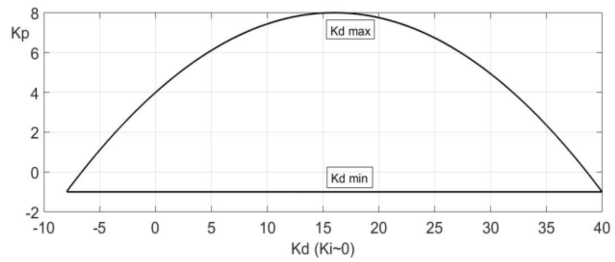

Figure 12. 2D Stability boundary curves for $k_p = f(k_d)$, k_i=0

If k_i=0, then $k_d^{min} = -8$<k_d<40 and $k_p^{min} = -1$<k_p<8 (Fig.11). At the left intersection of lines ($k_d^{min} = -8$ and $k_p^{min} = -1$), the roots of the closed-loop system are: -1.0000 + 0.7071i;

-1.0000 - 0.7071i; 0.0000 + 0.0000i; -0.0000 - 0.0000i; -0.0000 - 0.0000i. At the right intersection of lines ($k_d^{max} = 40$ and $k_p^{min} = -1$), the roots of the closed − loop system are: -2 + 0.0000i; -0.0000 + 1.2246i; -0.0000 - 1.2246i; -0.0000 + 0.0000i; -0.0000 + 0.0000i. When k_i reaches its maximum value (k_i^{max} =18), the plots on Fig. 9 and Fig. 10 are converged to a single point and other gains reach their single maximum values, i.e. $k_p^{max} = 8$, $k_d^{max} = 40$. The roots of the system characteristic polynomial at this point are -2.0000 + 0.0000i; -0.0000 + 0.8660i; -0.0000 - 0.8660i; 0.0000 + 0.8660i; 0.0000 - 0.8660i, i.e., the system has double conjugate roots on imaginary axis of s-plane.

Conclusions. The paper presents an effective and simple tool for analytical solution of stability problem of higher-order linear time-invariant dynamic systems. It has a major advantage compared to Routh–Hurwitz technique. The proposed universal stability criteria (2) or (3) establish unique relations between the stability boundary values of the system characteristic polynomial coefficients and the newly introduced additional parameter k. It is a new approach and there are no similarities found to these criteria in the literature. The newly-developed method is a universal one and can be applied to any higher-order dynamic system. The authors of this paper have discovered and established a set of general expressions (2) or (3) that can be applied for derivation of necessary stability criteria for any order linear time-invariant dynamic system. The single analytical expressions of the non-zero coefficients of the system characteristic polynomial at its stability boundary conditions have been derived in section III for the systems with orders from 3 to 6 and successfully tested by using MATLAB software for numerous examples. The method has also been extended to prove numerically the stability boundaries problem solution for any system with order higher than six in section IV for a randomly selected example of eleventh-order system. The stability range of values for one of the coefficients of the characteristic equation for each of the dynamic system with orders from three to six has been derived analytically. The numerical example of eleventh order system is presented in the paper to prove validity of *fundamental law of marginal or boundary stability* for systems with orders higher than six. As a unique achievement, the marginal stability conditions for dynamic systems with *possible zero coefficients* and with multiple roots on the $j\omega$-axis of the s-plane have also been discussed in details. These results are new and have not been published currently in the literature and were obtained for special cases of marginal stability when the same exactly set of zero coefficients the system can be either in the state of *marginal stability* or *marginal instability*, i.e. the system exhibits a *dual behaviour*. Section V is dedicated to use of criteria (2) and (3) to provide marginal and absolute stability for the closed-loop control systems with proportional, derivative and integral gains. The paper discusses in detail the derivation of equations for precise stability boundary values of k_p, k_d, k_i gains based on the two-polynomial criteria (2) and (3). The obtained results of analytical calculation of precision stability boundary values for a multiple-gain higher-order closed-loop system do not have analogy currently in the control theory. The results obtained in this paper prove that the developed system stability criteria or algorithm for stability analysis of a higher-order linear dynamic system is a step forward in analysing stability conditions of complex dynamic systems and deriving precise analytical expressions for multiple gains of closed-loop control systems. This method is successfully tested on the *model of hard disk drive* with single-gain lead compensator [19] and on the *model of two-inertia system* with multiple gain controller design [20].

REFERENCES

[1]. N. S. Nise. *Transient response stability*. Control Syst. Eng., 5th Int. Student ed. Singapore: Wiley, 2008, ch. 6, sec. 2–4, pp. 291–306.
[2]. R. C. Dorf and R. H. Bishop. *The stability of linear feedback systems.* Modern Control Systems, 12th ed. Prentice–Hall, 2011, ch. 6, sec. 2, pp. 391–398.
[3]. M. Gopal. *Concept of stability and the Ruth Stability Criterion.* Control Systems, Principles and Design, 2nd ed. McGraw Hill,2003, ch.5, sec. 4, pp. 314-326.
[4]. B. Kuo and M. F. Golnaraghi. *Stability of linear control systems.* Automatic Control Systems, 7th ed. Englewood Cliffs, NJ: Prentice-Hall, 1995, ch. 6, sec. 5, pp. 334–343.
[5]. M. Ho, A. Datta, and S. P. Bhattacharyya. *An elementary derivation of the Routh-Hurwitz criterion*. IEEE Trans. Autom. Control, vol. 43, no. 3, pp. 405–409, Mar. 1998.
[6]. M. M. Fahmy and J. O'Reilly. *A note on the Routh-Hurwitz test*. IEEE Trans. Autom. Control, vol. AC-27, no. 2, pp. 483–485, Apr.1982.
[7]. K. S. Yeung. *Routh-Hurwitz test under vanishing leading array elements*. IEEE Trans. Autom. Control, vol. AC-28, no. 1, pp. 104–106, Jan. 1983.
[8]. K. Khatwani. *On Routh-Hurwitz criterion.* IEEE Transactions on Automatic Control, vol. 26, no. 2, pp. 583-584, Apr 1981.

[9]. S. Pillai. *The ε method of the Routh-Hurwitz criterion. IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 584-584, Apr 1981.
[10]. S. S. Chen and J. S. H. Tsai. *On the singular cases of the complex Routh algorithm for stability tests*. IMA J. Math. Contr. Inform., vol. 10, pp. 71–82, 1993.
[11]. R. N. Clark. *The Routh-Hurwitz stability criterion, revisited. IEEE Control Systems*, vol. 12, no. 3, pp. 119-120, Jun 1992.
[12]. M. A. Choghadi and H. A. Talebi. *The Routh-Hurwitz Stability Criterion, Revisited: The Case of Multiple Poles on Imaginary Axis.* IEEE Transactions on Automatic Control, Vol. 58, No. 7, July 2013.
[13]. B.B. Alagoz, *On the Harmonic Oscillation of High-order Linear Time Invariant Systems.* Computer Science, Discrete Mathematics, arXiv: 1403. 2170, pp.1-12, 2014.
[14]. N. Tan, "*Computation of Stabilizing PI and PID controllers for processes with time delay*", ISA Transactions, 44, pp 213-223, 2005.
[15]. T. V. Moghaddam T.V. and Abbasi Y. *Tuning a Fractional Order PD and PID Controller with Lead Compensator for Integrating Time Delay Systems.* Journal of Electrical and Control Engineering, vol. 2, pp 34-41, 2012.
[16]. J. Chen, P. Fu. *Stability Analysis of Polynomially Dependent Systems by Eigenvalue Perturbation.* IEEE Transactions on Automatic Control, vol. 62, issue 11, pp. 5915-5922, 2017.
[17]. L. Wang, B. K. P. Horn, G. Strang. *Eigenvalue and Eigenvector Analysis of Stability for a Line of Traffic.* Studies in Applied Mathematics Journal, vol. 138, issue 1, pp. 103-132, 2017.
[18]. N. Mir-Nasiri, M. H. Ali. *A New Algorithm to Control Dynamic Stability of Higher-Order Systems.* Proceedings of IEEE International Conference on Control Systems, Computing and Engineering, 23-25 November, Penang, Malaysia, 2018.
[19]. A. Nath, S. Kaitwanidvilai. *High performance HDD servo system using GA based fixed structure robust loop Shaping control.* Proceedings of IEEE International Conference on Robotics and Biomimetics (ROBIO), 19-23 December, Guilin, China, pp. 1854-1859, 2009.
[20]. G. Zhang, J. Furusho. *Speed Control of Two-Inertia System by PI/PID Control.* IEEE Transactions on Industrial Electronics, vol. 47, issue 3, pp. 603-609, 2000.

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